Sheaves of C^*-Algebras

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Dedicated to the memory of John Dauns, 24 June 1937 – 4 June 2009 joint work in progress with Pere Ara (Barcelona)

based on

P. Ara and M. Mathieu, Local multipliers of C^*-algebras, Springer-Verlag, London, 2003.

P. Ara and M. Mathieu, *A not so simple local multiplier algebra*, J. Funct. Analysis **237** (2006), 721–737.

P. Ara and M. Mathieu, *Maximal C^*-algebras of quotients and injective envelopes of C^*-algebras*, Houston J. Math. **34** (2008), 827–872.

Bundles of C^*-algebras

Definition

For a topological space X, an *upper semicontinuous C^*-bundle over X* (in short, a *usc C^*-bundle over X*) is a triple (A, π, X) consisting of a topological space A and an open, continuous surjection $\pi\colon \mathsf{A} \to X$ with each fibre $\mathsf{A}_x := \pi^{-1}(x)$ a C^*-algebra and such that the function $\|\cdot\|\colon \mathsf{A} \to \mathbb{R}$ defined by $a \mapsto \|a\|_{\mathsf{A}_{\pi(a)}}$ is upper semicontinuous and all algebraic operations are continuous on A;
that is, $+$ and $\cdot$ are continuous functions $\mathsf{A} \times_\pi \mathsf{A} \to \mathsf{A}$ (where $\mathsf{A} \times_\pi \mathsf{A} = \{(a_1, a_2) \in \mathsf{A} \times \mathsf{A} \mid \pi(a_1) = \pi(a_2)\}$) and ${}^*\colon \mathsf{A} \to \mathsf{A}$ as well as $\cdot_{\mathbb{C}}\colon \mathbb{C} \times \mathsf{A} \to \mathsf{A}$ are continuous.

Bundles of C^*-algebras

Definition (ctd.)

Denoting by $\Gamma_b(U, \mathsf{A})$, $U \in \mathcal{O}_X$ the set of all bounded continuous sections $s\colon U \to \mathsf{A}$ of π we further require the following properties.

(i) For all $U \in \mathcal{O}_X$, $s \in \Gamma_b(U, \mathsf{A})$ and $\varepsilon > 0$, the set

$$V(U, s, \varepsilon) := \{a \in \mathsf{A} \mid \pi(a) \in U \text{ and } \|a - s(\pi(a))\| < \varepsilon\}$$

is an open subset of A and these sets form a basis for the topology of A.

(ii) For each $x \in X$, we have

$$\mathsf{A}_x = \overline{\{s(x) \mid s \in \Gamma_b(U, \mathsf{A}),\ U \text{ an open neighbourhood of } x\}}.$$

Bundles of C^*-algebras

Example

$A = C(X, B(H))$ yields a trivial continuous C^*-bundle over the compact Hausdorff space X with each fibre equal to $B(H)$.

Example (Somerset)

For a separable unital C^*-algebra A, $M_{\mathrm{loc}}(A)$ can be realised as a continuous C^*-bundle over $\mathrm{Glimm}(M_{\mathrm{loc}}(A)) = \beta\,\mathrm{Prim}(M_{\mathrm{loc}}(A))$, the Glimm ideal space of $M_{\mathrm{loc}}(A)$, with all fibres being primitive C^*-algebras.

Bundles of C^*-algebras

X a locally compact Hausdorff space

Definition

A C^*-algebra A is a *$C_0(X)$-algebra* if there is an essential $*$-homomorphism $\iota\colon C_0(X) \to ZM(A)$ (i.e., $\overline{\iota(C_0(X))A} = A$).

Definition

A *C^*-algebra over X* is a pair (A, ψ) consisting of a C^*-algebra A and a continuous mapping $\psi\colon \mathrm{Prim}(A) \to X$.

Bundles of C^*-algebras

X a locally compact Hausdorff space

Theorem (Fell, Lee)

For a C^*-algebra A, the following conditions are equivalent:

(a) *A is a $C_0(X)$-algebra;*

(b) *(A, ψ) is a C^*-algebra over X;*

(c) *A is the section algebra of a usc C^*-bundle (A, π, X) (that is, there is a $C_0(X)$-linear isomorphism from A onto $\Gamma_0(X)$) .*

Moreover, (A, π, X) is a continuous C^*-bundle if and only if $\psi \colon \mathrm{Prim}(A) \to X$ is open.

Sheaves of C^*-algebras

X a topological space;
$\mathcal{O}_X$ category of open subsets (with open subsets U as objects and $V \to U$ if and only if $V \subseteq U$).

$\mathcal{C}^*$ category of C^*-algebras.

Definition

A *presheaf of C^*-algebras* is a contravariant functor $\mathfrak{A}\colon \mathcal{O}_X \to \mathcal{C}^*$.

A *sheaf of C^*-algebras* is a presheaf $\mathfrak{A}$ such that $\mathfrak{A}(\emptyset) = 0$ and, for every open subset U of X and every open cover $U = \bigcup_i U_i$, the maps $\mathfrak{A}(U) \to \mathfrak{A}(U_i)$ are the limit of the diagrams $\mathfrak{A}(U_i) \to \mathfrak{A}(U_i \cap U_j)$ for all i, j.

Sheaves of C^*-algebras

Universal Property:

$$\begin{array}{ccccc} \mathfrak{A}(U) & \xrightarrow{\rho} & \prod_i \mathfrak{A}(U_i) & \overset{\nu}{\underset{\mu}{\rightrightarrows}} & \prod_{i,j} \mathfrak{A}(U_i \cap U_j) \\ \uparrow & \nearrow_{\sigma} & & & \\ B & & & & \end{array}$$

$U_i \cap U_j \longrightarrow U_i$ yields $\rho_{ji} \colon \mathfrak{A}(U_i) \longrightarrow \mathfrak{A}(U_i \cap U_j)$;

similarly, $\rho_i \colon \mathfrak{A}(U) \longrightarrow \mathfrak{A}(U_i)$

requirement $\nu \circ \rho = \mu \circ \rho$;

if (B, σ) has like properties as $(\mathfrak{A}(U), \rho)$ then $\exists!\ B \longrightarrow \mathfrak{A}(U)$.

Sheaves of C^*-algebras

Notation and Terminology:

the C^*-algebra $\mathfrak{A}(U)$ is the *section algebra* over $U \in \mathcal{O}_X$;

by $s_{|V}$, $V \subseteq U$ open, we mean the "restriction" of $s \in \mathfrak{A}(U)$ to V; i.e., the image of s under $\mathfrak{A}(U) \to \mathfrak{A}(V)$;

the unique gluing property of a sheaf can be expressed as follows:

for each bounded compatible family of sections $s_i \in \mathfrak{A}(U_i)$, i.e., $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all i, j, there is a unique section $s \in \mathfrak{A}(U)$ such that $s_{|U_i} = s_i$ for all i.

Sheaves of C^*-algebras

Example 1. *Sheaves from bundles*

Let (A, π, X) be a usc C^*-bundle. Then

$$\Gamma_b(-, \mathsf{A})\colon \mathcal{O}_X \to \mathcal{C}_1^*, \quad U \mapsto \Gamma_b(U, \mathsf{A})$$

defines the *sheaf of bounded continuous local sections of* A, where $\mathcal{C}_1^*$ is the category of unital C^*-algebras.

$\Gamma_b(U, \mathsf{A}) \to \Gamma_b(V, \mathsf{A})$, $V \subseteq U$, is the usual restriction map.

Sheaves of C^*-algebras

Example 2. *The multiplier sheaf*

A C^*-algebra with primitive ideal space $\mathrm{Prim}(A)$;

$$\mathfrak{M}_A \colon \mathcal{O}_{\mathrm{Prim}(A)} \to \mathcal{C}_1^*, \quad \mathfrak{M}_A(U) = M(A(U)),$$

where $M(A(U))$ denotes the multiplier algebra of the closed ideal $A(U)$ of A associated to the open subset $U \subseteq \mathrm{Prim}(A)$.
$M(A(U)) \to M(A(V))$, $V \subseteq U$, the restriction homomorphisms.

Proposition

The above functor $\mathfrak{M}_A$ defines a sheaf of C^*-algebras.

Sheaves of C^*-algebras

Example 3. *The injective envelope sheaf*

let $I(B)$ denote the *injective envelope* of B;

$$\mathfrak{I}_A\colon \mathcal{O}_{\mathrm{Prim}(A)} \to \mathcal{C}_1^*, \quad \mathfrak{I}_A(U) = p_U I(A) = I(A(U)),$$

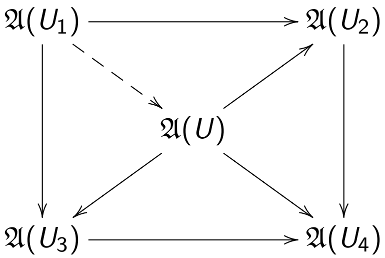
where $p_U = p_{A(U)}$ denotes the unique central open projection in $I(A)$ such that $p_{A(U)} I(A)$ is the injective envelope of $A(U)$. $I(A(U)) \to I(A(V))$, $V \subseteq U$, given by multiplication by p_V (as $p_V \leq p_U$).

$\{p_U \mid U \in \mathcal{O}_{\mathrm{Prim}(A)}\}$ is a complete Boolean algebra isomorphic to the Boolean algebra of regular open subsets of $\mathrm{Prim}(A)$, and it is precisely the set of projections of the AW^*-algebra $Z(I(A))$.

Sheaves of C^*-algebras

Example 4. *Sheaves over Alexandrov spaces*

X an Alexandrov space (i.e., each point has a smallest neighbourhood)
e.g., every finite topological space; highly non-Hausdorff



$X = \{x_1, \dots, x_4\} = U_1$
$U_2 = \{x_2, x_4\}$
$U_3 = \{x_3, x_4\}$
$U_4 = \{x_4\}$

Sheaves of C^*-algebras

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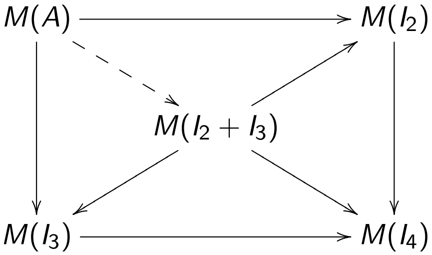

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Sheaves of C^*-algebras

Example 5. *Direct image functor*

let A be a C^*-algebra over X, i.e., a continuous mapping $\psi\colon \mathrm{Prim}(A) \to X$ is given;

let $\mathfrak{A}$ be a sheaf over $\mathrm{Prim}(A)$;

then $\psi_*(\mathfrak{A})$ defined by

$$\psi_*(\mathfrak{A})(U) = \mathfrak{A}(\psi^{-1}(U)) \qquad (U \in \mathcal{O}_X)$$

provides us with a new sheaf of C^*-algebras over X.

from sheaves to bundles

Theorem

Given a presheaf $\mathfrak{A}$ of C-algebras over X, there is a canonically associated upper semicontinuous C*-bundle (A, π, X) over X.*

Idea:

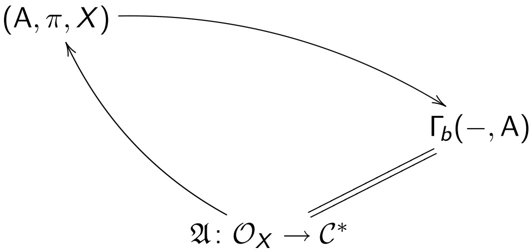
$x \in X$, define $\mathrm{A}_x := \varinjlim_{x \in U} \mathfrak{A}(U)$ (stalk at x)

let $\mathrm{A} := \bigsqcup_{x \in X} \mathrm{A}_x$ and define a topology on A by

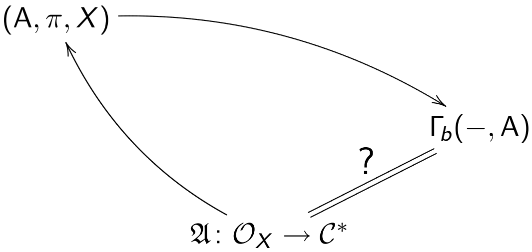
$$V(U, s, \varepsilon) = \{a \in \mathrm{A} \mid \pi(a) \in U \text{ and } \|a - s(\pi(a))\| < \varepsilon\}$$

is a basic open set, where $\varepsilon > 0$, $U \in \mathcal{O}_X$, $s \in \mathfrak{A}(U)$ and $s(x)$ the image under $\mathfrak{A}(U) \to \mathrm{A}_x$.

from bundles to sheaves



from sheaves to bundles and back?



from sheaves to bundles, and back

let $\mathfrak{A}$ sheaf of C^*-algebras over X, $U \in \mathcal{O}_X$

$$\mu_U \colon \mathfrak{A}(U) \to \Gamma_b(U, \mathrm{A}) \quad \text{injective *-homomorphism}$$
$$\mu_U(s)(x) = s(x) \qquad (s \in \mathfrak{A}(U),\ x \in U),$$

where $s(x)$ is the image under $\mathfrak{A}(U) \to \mathrm{A}_x = \varinjlim_{x \in V} \mathfrak{A}(V)$;

μ_U may be not surjective;

necessary condition: $\mathfrak{A}(U)$ is $C_b(U)$-module

from sheaves to bundles, and back

Definition

Let X be a topological space. The sheaf $\mathfrak{C}(X)$ of unital C^*-algebras over X is given by $\mathfrak{C}(U) = C_b(U)$, $U \in \mathcal{O}_X$ and the evident restriction mappings.

Therefore, if X is locally compact Hausdorff, $\mathfrak{C}(X)$ is nothing but the multiplier sheaf over X.

Proposition

Let X be a second countable, locally compact Hausdorff space and $\mathfrak{A}$ be a $\mathfrak{C}(X)$-sheaf of unital C^*-algebras over X. Then the maps $\mu_U \colon \mathfrak{A}(U) \to \Gamma_b(U, \mathrm{A})$ are isomorphisms for all $U \in \mathcal{O}_X$.

from sheaves to bundles, and back

Proposition

Let $\mathfrak{A}$ be a sheaf of C-algebras over an Alexandrov space X. Then the natural map $\mu_U \colon \mathfrak{A}(U) \to \Gamma_b(U, \mathsf{A})$ is an isomorphism for every open subset U of X.*

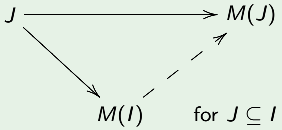
Local multipliers

Definition

For every C^*-algebra A,

$$M_{\mathrm{loc}}(A) = \varinjlim_{I \in \mathcal{I}_{ce}} M(I),$$

is its *local multiplier algebra*, where



$\mathcal{I}_{ce}$ the filter of all closed essential ideals of A.

Local multipliers

Pedersen's Question (1978):

Is $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$ for every C^*-algebra A?

A commutative:

$$M_{\mathrm{loc}}(A) = \varinjlim_{U\in\mathcal{D}} C_b(U) = \operatorname{alg}\varinjlim_{T\in\mathcal{T}} C_b(T) = I(A),$$

where $\mathcal{D}$ dense open; $\mathcal{T}$ dense G_δ subsets of $\mathrm{Prim}(A)$. Hence

$M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(I(A)) = I(A) = M_{\mathrm{loc}}(A)$.

A non-commutative, e.g., $A = C(X, B(H))$:

$$M_{\mathrm{loc}}(A) = \varinjlim_{U\in\mathcal{D}} C_b(U, B(H)_\beta) \subseteq \operatorname{alg}\varinjlim_{T\in\mathcal{T}} C_b(T, B(H)_\beta) = I(A),$$

where $\mathcal{D}$ dense open; $\mathcal{T}$ dense G_δ subsets of Stonean space X.

Depending on properties of X, $\subseteq$ can be strict and still $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = I(A)$!

The local multiplier sheaf

Definition

For a C^*-algebra A define the local multiplier sheaf $\mathfrak{Mloc}_A$ by

$$\mathfrak{Mloc}_A(U) = M_{\mathrm{loc}}(A(U)) = p_U M_{\mathrm{loc}}(A) \quad (U \in \mathcal{O}_{\mathrm{Prim}(A)}),$$

where $M_{\mathrm{loc}}(A) \subseteq I(A)$ and $p_U \in Z(M_{\mathrm{loc}}(A)) = Z(I(A))$.

note: $\mathfrak{M}_A \hookrightarrow \mathfrak{Mloc}_A \hookrightarrow \mathfrak{I}_A$ as sheaves

aim: *a sheaf representation of* $M_{\mathrm{loc}}(A)$

The derived sheaf of a presheaf

X Baire space (e.g., $X = \mathrm{Prim}(A)$)

$\mathcal{T}$ the family of dense G_δ's of X

(A, π, X) an upper semicontinuous C^*-bundle

$U \in \mathcal{O}_X$: $\mathfrak{D}(U) = \mathrm{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(T \cap U, \mathsf{A})$

$T' \subseteq T \in \mathcal{T}$: $\Gamma_b(T \cap U, \mathsf{A}) \to \Gamma_b(T' \cap U, \mathsf{A})$ restriction maps

Proposition

$\mathfrak{D} = \mathfrak{D}_{(\mathsf{A},\pi,X)}$ *is a presheaf of C^*-algebras over X.*

The derived sheaf of a presheaf

Definition

Let $\mathfrak{A}$ be a presheaf of C^*-algebras over a Baire space X. The *derived presheaf* $\mathfrak{D}_{\mathfrak{A}}$ of $\mathfrak{A}$ is the presheaf $\mathfrak{D}_{(\mathsf{A},\pi,X)}$.

Theorem

Let X be a Baire space. The map $\mathfrak{D}$ defines a functor

$$\mathfrak{D}\colon \mathcal{PSh}(X,\mathcal{C}_1^*) \longrightarrow \mathcal{Sh}(X,\mathcal{C}_1^*).$$

If $\iota\colon \mathfrak{A} \to \mathfrak{B}$ is a faithful natural transformation (that is, $\iota_U\colon \mathfrak{A}(U) \to \mathfrak{B}(U)$ is injective for every $U \in \mathcal{O}_X$), then $\mathfrak{D}(\iota)\colon \mathfrak{D}_{\mathfrak{A}} \to \mathfrak{D}_{\mathfrak{B}}$ is also faithful. For every presheaf $\mathfrak{A}$ of unital C^*-algebras over X, the sheaf $\mathfrak{D}_{\mathfrak{A}}$ is a $\mathfrak{D}_{\mathfrak{C}(X)}$-sheaf.

The derived sheaf of a presheaf

Theorem

For every C-algebra A, we have*

$$\mathfrak{D}_{\mathfrak{M}_A} \cong \mathfrak{Mloc}_A \quad \textit{and} \quad \mathfrak{D}_{\mathfrak{I}_A} \cong \mathfrak{I}_A$$

as sheaves over $\mathrm{Prim}(A)$.

hence

$$\mathfrak{Mloc}_A(U) = \operatorname{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(U\cap T, \mathsf{A}_{\mathfrak{M}_A}) \hookrightarrow \operatorname{alg}\varinjlim_{T\in\mathcal{T}} \Gamma_b(U\cap T, \mathsf{A}_{\mathfrak{I}_A}) = \mathfrak{I}_A(U)$$

for each $U \in \mathrm{Prim}(A)$.