Generalising Uniform Algebras Over Complete Valued Fields

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Introduction and Definitions

Uniform algebras have been extensively investigated because of their importance in the theory of uniform approximation and as examples of complex Banach algebras. As enquiry broadens one may ask whether analogous algebras exist when a complete valued field F other than the complex numbers is used as the underlying field over which the algebra is a vector space. To establish the context of our discussion we make the following definition.

Definition 1.0

Let F and L be complete valued fields such that L is an extension of F as a valued field. Let X be a compact Hausdorff space and let $C_L(X)$ be the Banach algebra of all continuous L-valued functions on X with pointwise operations and the sup norm. If a subset A of $C_L(X)$ satisfies:

(i) A is closed under pointwise operations
(ii) A is complete with respect to $\|\cdot\|_\infty$
(iii) $F \subset A$
(iv) A separates the points of X,

then we will call A an $^{L}/_{F}$ *uniform algebra* or just a *uniform algebra* when convenient.

In the language of Definition 1.0, an $^{L}/_{F}$ uniform algebra is an F Banach algebra of L-valued functions. Complex uniform algebras of the classical theory are $^{\mathbb{C}}/_{\mathbb{C}}$ uniform algebras. We now generalise two definitions by Kulkarni and Limaye from the theory of real function algebras.

Definition 1.1 (J. Mason 2009)

Let F and L be complete valued fields such that L is a finite extension of F as a valued field. If L is $\mathbb{R}$ or $\mathbb{C}$ then let X be a compact Hausdorff space, else let X be a compact subset of L. Define,

$$C(X,\tau,g) := \{f \in C_L(X) : f(\tau(x)) = g(f(x)) \text{ for all } x \in X\}$$

where: (i) $g \in \mathrm{Gal}(^{L}/_{F})$
(ii) $\tau : X \to X$ with $\mathrm{ord}(\tau)|\mathrm{ord}(g)$
(iii) g and τ are continuous.

We will call $C(X,\tau,g)$ the *basic* $^{L}/_{L^g}$ *function algebra on* (X,τ,g), where $L^g := \{x \in L : g(x) = x\}$, or just a *basic function algebra* when convenient.

Definition 1.2 (J. Mason 2009)

Let F and L be complete valued fields such that L is a finite extension of F as a valued field. Let (X,τ,g) conform to the conditions of Definition 1.1 and let A be a subset of the basic $^{L}/_{L^g}$ function algebra on (X,τ,g).

If A is also an $^{L}/_{L^g}$ uniform algebra then we will call A an $^{L}/_{L^g}$ *function algebra on* (X,τ,g).

Remarks:

(1) With respect to the above definitions the basic $^{L}/_{L^g}$ function algebra on (X,τ,g) is always an $^{L}/_{L^g}$ uniform algebra. In fact $\mathrm{ord}(\tau)|\mathrm{ord}(g)$ is an optimum condition since if we do not include it in Definition 1.1 then $C(X,\tau,g)$ separates the points of X if and only if $\mathrm{ord}(\tau)|\mathrm{ord}(g)$.

(2) In Definition 1.1, the continuity of g is a mild condition since g will automatically be continuous when convergence in L is coordinatewise convergence, viewing L as a finite dimensional vector space over F.

(3) In Definition 1.1 X is required to be a subset of L if L is not $\mathbb{R}$ or $\mathbb{C}$. This condition may be stronger than what is needed for Remark 1 to hold. However we can not always let X be any compact Hausdorff space. The field of p-adic numbers $\mathbb{Q}_p$ is an example of a nonarchimedean field and every nonarchimedean field is totally disconnected. Hence $C_{\mathbb{Q}_p}([0,1])$ only contains the constants.

(4) Not every $^{L}/_{L^g}$ function algebra on (X,τ,g) is basic. Let X be a compact subset of $\mathbb{C}$ with non empty interior. The $^{\mathbb{C}}/_{\mathbb{C}}$ uniform algebra $A(X)$ of all functions in $C_{\mathbb{C}}(X)$ that are analytic on the interior of X is not the basic $^{\mathbb{C}}/_{\mathbb{C}}$ function algebra on $(X,\mathrm{id},\mathrm{id})$, since this is $C_{\mathbb{C}}(X)$.

It is informative to put existing theorems from the theory of real function algebras in to the terms of these new definitions and then to consider whether they generalise to other complete valued fields. Note that a real function algebra is precisely a $^{\mathbb{C}}/_{\mathbb{R}}$ function algebra on $(X,\tau,\bar{z})$ for some X and τ where $\bar{z}$ is complex conjugation.

Theorem 1.0 (S. Kulkarni, B. Limaye)

Let A be a real commutative Banach algebra with unit 1 such that $\|a\|^2 = \|a^2\|$ for all a in A.

Then A is isometrically isomorphic to a $^{\mathbb{C}}/_{\mathbb{R}}$ function algebra on $(X,\tau,\bar{z})$.

The possibility of generalising Theorem 1.0 will be investigated during my current program of research. However finding the appropriate field L could be tricky.

Fields and Examples

When contemplating examples there are some obvious choices of complete valued fields to use. Every field of characteristic zero contains the rational numbers $\mathbb{Q}$. We can choose any one of the completions of $\mathbb{Q}$ (given by Ostrowski's theorem) to play the role of F in Definition 1.1. Fields of characteristic p^q, where $p, q \in \mathbb{N}$ with p a prime, can also be used. The following table is not comprehensive. Note that a nonarchimedean field F is a complete field such that its valuation observes the strong triangle inequality, $|a+b| \leqslant \max\{|a|,|b|\}$ for all a, b in F.

Examples of complete valued fields available for Definition 1.1

Description	F	L	Valuation on L	Char(F) †	Type ‡
Classical fields	$\mathbb{R}$	$\mathbb{C}$	absolute	0	arch
p-adic numbers	$\mathbb{Q}_p$	$\mathbb{Q}_p(\alpha_1,\cdots,\alpha_n)$	extension of p-adic	0	nonarch
Number fields	$\mathbb{Q}$	$\mathbb{Q}(\alpha_1,\cdots,\alpha_n)$	trivial	0	nonarch
Finite fields	$\mathbb{F}_{p^q}$	$\mathbb{F}_{p^q}(\alpha_1,\cdots,\alpha_n)$	trivial	p	nonarch

† Characteristic of F, ‡ archimedean or nonarchimedean.

Definition 1.1 excludes L from being an infinite extension of F but the possibility of permitting certain infinite extension can be investigated. For example, with additional constraints, applying Definition 1.1 to each element in a countable tower of finite extensions of F and then taking L to be the completion of the union of these finite extensions might be a possibility. Before we look at some examples of basic function algebras we should be aware of the following generalisation of the Stone-Weierstrass theorem. Note that an ultrametric space (X,d) is a metric space such that d observes the strong triangle inequality, $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}$ for all x, y, z in X.

Theorem 1.1 (With reference to W. H. Schikhof)

Let $L \in \{\mathbb{R},\mathbb{C}\} \cup \{\mathbb{K} : \mathbb{K}$ is a complete nonarchimedean field$\}$. Let X conform to Definition 1.1 and let A be an $^{L}/_{L}$ function algebra on $(X,\mathrm{id},\mathrm{id})$.

Then either $A = C_L(X)$ or $L = \mathbb{C}$ and A is not self adjoint, that is there is $f \in A$ with $\bar{f} \notin A$.

For L a complete nonarchimedean field, Theorem 1.1 also holds when X is taken to be any compact ultrametric space. We will now look at some examples of basic function algebras.

(E1) Let $F = \mathbb{Q}_5$, $L = \mathbb{Q}_5(\sqrt{2})$ with the usual extension of the 5-adic valuation, $X := \{x \in L : |x|_L \leqslant 1\}$ and let g be the Galois automorphism that sends $\sqrt{2}$ to $-\sqrt{2}$. It turns out that g is an isometry on L so we can take $\tau = g$. In this case $C(X,\tau,g)$ has the property that every power series in $C(X,\tau,g)$ has $\mathbb{Q}_5$ valued coefficients. However, since $X \subset \mathbb{Q}_5(\sqrt{2})$ these power series are $\mathbb{Q}_5(\sqrt{2})$ valued functions.

(E2) Let F, L, X and g be as in E1. We can obtain a function $\omega : L \to \mathbb{Z} \cup \{+\infty\}$ such that for all $x \in L$ we have $|x|_L = 5^{-\omega(x)}$. Define $\tau(0) = 0$ and for $x \in X\backslash\{0\}$,

$$\tau(x) := \begin{cases} 5x & \text{if } \; 2 \mid \omega(x) \\ 5^{-1}x & \text{if } \; 2 \nmid \omega(x). \end{cases}$$

In this case the only power series in $C(X,\tau,g)$ are constants belonging to $\mathbb{Q}_5$.

References

(1) S. H. Kulkarni and B. V. Limaye, *Real function algebras*, Monographs and textbooks in pure and applied mathematics, no. 168, Marcel Dekker inc, 1992.

(2) W. H. Schikhof, *Ultrametric calculus an introduction to p-adic analysis*, Cambridge University Press, 2006.

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