When James met Schreier

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Outline

We amalgamate two important classical examples of Banach spaces:

- James' quasi-reflexive Banach spaces, and
- Schreier's space giving a counterexample to the Banach–Saks property,

to obtain a family of *James–Schreier spaces*. We then investigate their properties.

Key point: like the James spaces, each James–Schreier space is a commutative Banach algebra with a bounded approximate identity when equipped with the pointwise product.

Note: this is joint work with Alistair Bird who will cover further aspects of it in his talk.

James' quasi-reflexive Banach space J — motivation

- Defined by James in 1950–51.
- Key property: J is *quasi-reflexive:*

$$\dim J^{**}/\kappa(J) = 1,$$

  where $\kappa\colon J \to J^{**}$ is the canonical embedding.
- Resolved two major open problems:
  - a Banach space with separable bidual need not be reflexive;
  - a separable Banach space which is (isometrically) isomorphic to its bidual need not be reflexive.
- Subsequently, many other interesting properties have been added to this list, for instance:
  - Bessaga & Pełczyński (1960): An infinite-dimensional Banach space X need not be isomorphic to its Cartesian square $X \oplus X$.
  - Edelstein–Mityagin (1970): There may be *characters* on the Banach algebra $\mathscr{B}(X)$ of bounded linear operators on an infinite-dimensional Banach space X. Specifically, the quasi-reflexivity of J implies that

$$\dim \mathscr{B}(J)/\mathscr{W}(J) = 1,$$

    where $\mathscr{W}(J)$ is the ideal of weakly compact operators on J.

Conventions

- Throughout, the scalar field is either $\mathbb{K} := \mathbb{R}$ or $\mathbb{K} := \mathbb{C}$.
- c_{00} denotes the vector space of all finitely supported sequences in $\mathbb{K}$:

$$c_{00} := \big\{(\alpha_n) : \alpha_n \in \mathbb{K}\ (n \in \mathbb{N}) \text{ and } \exists\, N \in \mathbb{N}\colon \alpha_n = 0\ (n \geqslant N)\big\}.$$

- For each $n \in \mathbb{N}$, let

$$e_n := (0, 0, \ldots, 0, \underset{\text{pos. } n}{1}, 0, 0, \ldots) \in c_{00};$$

  then $(e_n)_{n\in\mathbb{N}}$ is a vector-space basis for c_{00}.

Definition. For $1 \leqslant p < \infty$, $x = (\alpha_n) \in c_{00}$ and $A = (n_1, \ldots, n_{k+1})$, where $k, n_1, \ldots, n_{k+1} \in \mathbb{N}$ with $n_1 < n_2 < \cdots < n_{k+1}$, let

$$\mu_p(x, A) := \left(\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p \right)^{\frac{1}{p}}.$$

This defines a seminorm $\mu_p(\cdot, A)$ on c_{00}; taking the supremum over all such A, we obtain a norm — the p^{th} *James norm:*

$$\begin{aligned} \|x\|_{J_p} &:= \sup_A \mu_p(x, A) \\ &= \sup \Bigg\{ \left(\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p \right)^{\frac{1}{p}} : k, n_1, \ldots, n_{k+1} \in \mathbb{N}, \\ &\qquad\qquad n_1 < n_2 < \cdots < n_{k+1} \Bigg\}. \end{aligned}$$

The completion of c_{00} with respect to this norm is the p^{th} *James space* J_p.

Properties of the James spaces

- $J_1 \cong \ell_1$; for this reason, we shall only consider $p > 1$.

James originally considered only the case where $p = 2$. However, his proofs generalize easily to all $p > 1$, giving:

- J_p is *quasi-reflexive:*
$$\dim J_p^{**}/\kappa(J_p) = 1,$$
where $\kappa\colon J_p \to J_p^{**}$ is the canonical embedding;
- J_p is isomorphic to J_p^{**}; by modifying the definition of the norm given above slightly, James even obtained that J_p is *isometrically* isomorphic to J_p^{**} (and the modified norm is equivalent to the one defined above).

Further properties of the James spaces

- Herman & Whitley (1967; $p = 2$): J_p is ℓ_p-*saturated* — every closed, infinite-dimensional subspace of J_p contains a further subspace X which is isomorphic to ℓ_p.
- Casazza, Lin & Lohman (1977; $p = 2$): it is always possible to choose such an X with the additional property that X is complemented in J_p.
- Andrew & Green (1980; $p = 2$): J_p is a Banach algebra with respect to the pointwise product.

Definition. A sequence $(b_n)_{n\in\mathbb{N}}$ in a Banach space X is a *(Schauder) basis* for X if, for each $x \in X$, there is a unique sequence $(\alpha_n)_{n\in\mathbb{N}}$ of scalars such that the series

$$\sum_{n=1}^{\infty} \alpha_n b_n$$

is convergent with sum x.

Example: the unit vector basis for ℓ_p $(1 \leqslant p < \infty)$ and c_0

Let $X := \ell_p$ for some $p \in [1, \infty)$ or $X := c_0$, and recall that

$$e_n := (0, 0, \ldots, 0, \underset{\text{pos. } n}{1}, 0, \ldots) \qquad (n \in \mathbb{N}).$$

Claim. $(e_n)_{n \in \mathbb{N}}$ is a Schauder basis for X, called the *standard basis* or the *unit vector basis*.

This is proved by verifying the following two conditions:

(∃) for each $x = (\alpha_n) \in X$, the series $\sum_n \alpha_n e_n$ is convergent with sum x;

(!) if $\sum_n \alpha_n e_n = 0$, then $\alpha_n = 0$ for each $n \in \mathbb{N}$.

Similarly, $(e_n)_{n \in \mathbb{N}}$ is a Schauder basis for J_p for each $p \in (1, \infty)$.

The role of coordinate signs

There is one important difference between the two examples above:

Changing the signs of some coordinates does not change the norm of an element of ℓ_p (or c_0), that is, for any sequence (ε_n) of signs (meaning that $\varepsilon_n = \pm 1$ for each $n \in \mathbb{N}$), we have

$$\Big\|\sum_n \varepsilon_n \alpha_n e_n\Big\|_{\ell_p} = \Big(\sum_n |\alpha_n|^p\Big)^{\frac{1}{p}} = \Big\|\sum_n \alpha_n e_n\Big\|_{\ell_p}$$

for each $\sum_n \alpha_n e_n \in \ell_p$.

The role of coordinate signs (continued)

In contrast, changing signs can make the James norm ‘blow up’; for instance,

$$\chi_m := \sum_{j=1}^{m} e_j = (1, 1, \dots, \underset{\text{pos. } m}{1}, 0, 0, \dots)$$

is a unit vector in J_p (no matter what $m \in \mathbb{N}$ and $p \in (1, \infty)$ are), but if we change every other sign (and, for the sake of argument, let m be even),

$$\sum_{j=1}^{m} (-1)^j e_j = (-1, 1, -1, 1, \dots, -1, \underset{\text{pos. } m}{1}, 0, 0, \dots),$$

the resulting vector has norm

$$\big(|-1-1|^p + |1-(-1)|^p + \cdots + |-1-1|^p + |1-0|^p\big)^{\frac{1}{p}} = \big(2^p(m-1)+1\big)^{\frac{1}{p}}$$

which tends to ∞ as $m \to \infty$.

Theorem. For a sequence (x_n) in a Banach space, the following conditions are equivalent:

(a) for each sequence (ε_n) of signs, the series $\sum_n \varepsilon_n x_n$ is convergent;

(b) for each $(\alpha_n) \in \ell_\infty$, the series $\sum_n \alpha_n x_n$ is convergent;

(c) for each permutation π of $\mathbb{N}$, the series $\sum_n x_{\pi(n)}$ is convergent.

In the positive case, the series $\sum_n x_n$ is *unconditionally convergent*, and there is a constant $C \geqslant 1$ such that

$$\Big\| \sum_n \alpha_n x_n \Big\| \leqslant C \big\| (\alpha_n) \big\|_{\ell_\infty} \quad \big((\alpha_n) \in \ell_\infty \big).$$

Note. Many more equivalent conditions can be added to the list above!

Definition. A Schauder basis $(b_n)_{n\in\mathbb{N}}$ for a Banach space X is *unconditional* if, for each $x = \sum_n \alpha_n b_n \in X$, the series $\sum_n \alpha_n b_n$ converges unconditionally.

The examples above show that:

- $(e_n)_{n\in\mathbb{N}}$ is an unconditional Schauder basis for ℓ_p $(1 \leqslant p < \infty)$ and c_0;
- however, $(e_n)_{n\in\mathbb{N}}$ is *not* an unconditional Schauder basis for J_p for any $1 < p < \infty$.
  In fact, it can be shown that J_p does not have an unconditional Schauder basis and, more generally, does not embed in a Banach space with an unconditional Schauder basis.

Unconditional Schauder bases and Banach algebras

Let X be a Banach space with an unconditional Schauder basis $(b_n)_{n\in\mathbb{N}}$, and suppose that $(b_n)_{n\in\mathbb{N}}$ is *semi-normalized:*

$$\inf_n \|b_n\| > 0 \qquad \text{and} \qquad \sup_n \|b_n\| < \infty.$$

Then, for each $x = \sum_n \alpha_n b_n$ and $y = \sum_n \beta_n b_n$ in X, the series

$$xy := \sum_n \alpha_n \beta_n b_n$$

converges in X because $(\alpha_n) \in c_0 \subseteq \ell_\infty$. In other words, X is closed under the pointwise product (and this product is separately continuous). Hence, by passing to an equivalent norm, X becomes a Banach algebra with respect to this product; it is clearly commutative and non-unital.
In the case where $\mathbb{K} = \mathbb{C}$, pointwise complex conjugation

$$\Big(\sum_n \alpha_n b_n\Big)^* := \sum_n \overline{\alpha}_n b_n$$

defines a continuous involution on X.

Example. ℓ_p (for $1 \leqslant p < \infty$) and c_0 are Banach $*$-algebras with respect to the pointwise operations.

Although the James spaces J_p (for $p > 1$) do not have unconditional Schauder bases, Andrew & Green (1980; $p = 2$) showed that J_p is nevertheless a Banach $*$-algebra with respect to the pointwise operations (after passing to an equivalent norm). Sample results:

- J_p is semisimple;
- J_p^{**} is the unitization of J_p, as well as the multiplier algebra of J_p;
- $\chi_m := \sum_{j=1}^m e_j = (1, 1, \ldots, \underset{\text{pos. } m}{1}, 0, 0, \ldots)$ $(m \in \mathbb{N})$ defines a sequential bounded approximate identity for J_p contained in c_{00};
- the closed ideals in J_p are precisely the subspaces of the form

$$\overline{\operatorname{span}}\{e_n : n \in N\}$$

  for some subset $N \subseteq \mathbb{N}$;

Consequence. The basic amenability questions can be answered for J_p:

- J_p is weakly amenable, but not amenable (Dales (2000)/White (2003));
- J_p is sequentially approximately contractible (by a result of Ghahramani, Loy & Zhang (2008), using work of Dales, Loy & Zhang (2006)).

Banach & Saks (1930) proved that, for $1 < p < \infty$, each weakly convergent sequence (x_n) in $L_p[0,1]$ has a subsequence (x_{n_j}) such that the sequence of arithmetic means

$$\frac{1}{N}\sum_{j=1}^{N} x_{n_j} \qquad (N \in \mathbb{N})$$

converges in norm.

They went on to ask if this is also true in $C[0,1]$.

Note. By reflexivity of L_p, 'weakly convergent' can be replaced by the formally weaker assumption that (x_n) is bounded; this is the modern formulation of the Banach–Saks Theorem.

Schreier (1930) constructed an example answering Banach and Saks' question in the negative. His method led to the definition of the Banach space named after him: the *Schreier space.*

The key ingredient: admissible sets

Definition. A non-empty, finite subset A of $\mathbb{N}$ is *admissible* if

$$|A| \leqslant \min A.$$

In other words, writing $A = \{n_1 < n_2 < \cdots < n_k\}$, we have

$$A \text{ admissible} \iff k \leqslant n_1.$$

Example. Each of the following sets is admissible:

$$\{1\}, \{2,3\}, \{2,7\} \quad \text{and} \quad \{3,17,29\}.$$

The definition of the Schreier space

Definition. For $x = (\alpha_n) \in c_{00}$ and a non-empty subset A of $\mathbb{N}$, let

$$\nu_1(x, A) := \sum_{n \in A} |\alpha_n|.$$

This defines a seminorm $\nu_1(\cdot, A)$ on c_{00}; taking supremum over all admissible sets A, we obtain a norm — the *Schreier norm:*

$$\begin{aligned}\|x\|_{S_1} &:= \sup\{\nu_1(x, A) : A \subseteq \mathbb{N} \text{ admissible}\}\\ &= \sup\Big\{\sum_{j=1}^{k} |\alpha_{n_j}| : 1 \leqslant k \leqslant n_1 < \cdots < n_k\Big\}.\end{aligned}$$

The completion of c_{00} with respect to this norm is the *Schreier space* S_1.

It was formally introduced by Beauzamy (1979), building on Baernstein's (1972) construction of a reflexive Banach space without the Banach–Saks property.

Definition. For $1 \leqslant p < \infty$, $x = (\alpha_n) \in c_{00}$ and a non-empty subset A of $\mathbb{N}$, let

$$\nu_p(x, A) := \Big(\sum_{n \in A} |\alpha_n|^p\Big)^{\frac{1}{p}}.$$

This defines a seminorm $\nu_p(\cdot, A)$ on c_{00}; taking supremum over all admissible sets A, we obtain a norm — the p^{th} *Schreier norm:*

$$\begin{aligned} \|x\|_{S_p} &:= \sup\big\{\nu_p(x, A) : A \subseteq \mathbb{N} \text{ admissible}\big\} \\ &= \sup\bigg\{\Big(\sum_{j=1}^{k} |\alpha_{n_j}|^p\Big)^{\frac{1}{p}} : 1 \leqslant k \leqslant n_1 < \cdots < n_k\bigg\}. \end{aligned}$$

The completion of c_{00} with respect to this norm is the p^{th} *Schreier space* S_p.

Properties of the Schreier spaces

Let $1 \leqslant p < \infty$. Then:

- $(e_n)_{n\in\mathbb{N}}$ is an unconditional Schauder basis for S_p.
  In particular, S_p is a commutative Banach $*$-algebra with respect to the pointwise operations.
- S_p is c_0-saturated: every closed, infinite-dimensional subspace of S_p contains a subspace isomorphic to c_0 (and this subspace is automatically complemented in S_p by Sobczyk's Theorem).
  In particular, ℓ_∞ embeds in S_p^{**}, so that S_p is not (quasi-)reflexive.
- S_p is isomorphic to its Cartesian square:

$$S_p \cong S_p \oplus S_p.$$

The definition of the p^{th} James–Schreier norm

We are now going to amalgamate the definitions of the p^{th} James norm and the p^{th} Schreier norm.
Recall that, for $1 \leqslant p < \infty$, $x = (\alpha_n) \in c_{00}$ and $A = (n_1, \dots, n_{k+1})$, where $k, n_1, \dots, n_{k+1} \in \mathbb{N}$ with $n_1 < n_2 < \cdots < n_{k+1}$, we have defined

$$\mu_p(x, A) := \left(\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p \right)^{\frac{1}{p}}.$$

The p^{th} James norm is obtained by taking the supremum over all such A.
If instead we restrict ourselves to consider only those A for which $k \leqslant n_1$ (the *permissible* sets), then we obtain the *p^{th} James–Schreier norm:*

$$\begin{aligned} \|x\|_{V_p} &:= \sup_A \mu_p(x, A) \\ &= \sup \left\{ \left(\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p \right)^{\frac{1}{p}} : k, n_1, \dots, n_{k+1} \in \mathbb{N}, \right. \\ &\qquad\qquad \left. k \leqslant n_1 < n_2 < \cdots < n_{k+1} \right\}. \end{aligned}$$

The completion of c_{00} with respect to this norm is the *p^{th} James–Schreier space V_p*.

Let $1 \leqslant p < \infty$.

Proposition. $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis for V_p.

It is not unconditional (for a similar reason to J_p); in fact, we have:

Theorem. V_p does not embed in a Banach space with an unconditional Schauder basis.

Corollary. V_p does not embed in S_q for any $q \in [1, \infty)$. In particular, $V_p \not\cong S_q$.

Proposition. Let $1 \leqslant p < \infty$. Then V_p is a Banach $*$-algebra with respect to the pointwise operations (after passing to an equivalent norm), and

- V_p is $*$-semisimple;
- $\chi_m := \sum_{j=1}^{m} e_j = (1, 1, \ldots, \underset{\text{pos. } m}{1}, 0, 0, \ldots)$ $(m \in \mathbb{N})$ defines a sequential bounded approximate identity for V_p contained in c_{00};
- V_p is sequentially approximately contractible and weakly amenable, but not amenable.

Theorem. V_p is c_0-saturated: every closed, infinite-dimensional subspace of V_p contains a subspace isomorphic to c_0 (and this subspace is automatically complemented in V_p by Sobczyk's Theorem).

Corollary. V_p does not embed in J_q for any $q \in (1, \infty)$ and, *vice versa*, J_q does not embed in V_p.

Block basic sequences

Definition. Let $(b_n)_{n\in\mathbb{N}}$ be a Schauder basis for a Banach space. A *block basic sequence* of $(b_n)_{n\in\mathbb{N}}$ is a sequence $(u_n)_{n\in\mathbb{N}}$ of non-zero vectors of the form

$$u_n = \sum_{j=M_n}^{N_n} \alpha_j b_j \qquad (n \in \mathbb{N}),$$

where $1 \leqslant M_1 \leqslant N_1 < M_2 \leqslant N_2 \cdots$ are integers and $\alpha_1, \alpha_2, \ldots$ scalars.

Picture. A block basic sequence of the unit vector basis might be depicted as

$$\begin{aligned} u_1 &= (*, *, *, 0, 0, \ldots) \\ u_2 &= (0, 0, 0, *, *, 0, \ldots) \\ u_3 &= (0, 0, 0, 0, 0, 0, *, *, *, *, 0, \ldots) \\ &\vdots \end{aligned}$$

Why are block basic sequences important?

Theorem. (Bessaga & Pełczyński (1958).) Let $(b_n)_{n\in\mathbb{N}}$ be a Schauder basis for a Banach space X. Then every closed, infinite-dimensional subspace Y of X contains a closed, infinite-dimensional subspace Z which is isomorphic to the closed linear span of a block basic sequence of $(b_n)_{n\in\mathbb{N}}$.

Outline of the proof that V_p is c_0-saturated

- Given a closed, infinite-dimensional subspace Y of V_p, there exists a subspace Z of Y which is isomorphic to the closed span of some block basic sequence $(u_n)_{n\in\mathbb{N}}$ of $(e_n)_{n\in\mathbb{N}}$.
- This reduces the problem to the following: Given a block basic sequence $(u_n)_{n\in\mathbb{N}}$ of $(e_n)_{n\in\mathbb{N}}$, show that $(u_n)_{n\in\mathbb{N}}$ has a block basic sequence $(v_n)_{n\in\mathbb{N}}$ which is equivalent to the standard basis for c_0.
  (*'Equivalent'* means that the linear mapping given by $v_n \mapsto e_n \in c_0$ for each $n \in \mathbb{N}$ is an isomorphism.)
- We achieve this by inductively constructing the blocks v_n such that on the one hand they are normalized:

$$\|v_n\|_{V_p} = 1 \qquad (n \in \mathbb{N}),$$

  and on the other they are very 'flat' in the following sense: suppose that $v_1, \dots, v_n$ have been chosen, and write

$$v_n = \sum_{j=M_n}^{N_n} \alpha_j e_j;$$

  then we want

$$\|v_{n+1}\|_{\ell_\infty} < N_n^{-\frac{1}{p}}.$$