Compact and weakly compact derivations from commutative Banach algebras.

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Derivations.

Let A be a complex, associative algebra and E and A-bimodule. A *derivation* from A to E is a linear map

$$D : A \longrightarrow E$$

such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A).$$

We shall concentrate on bounded derivations from Banach algebras into their duals.
For a Banach algebra A we call the topological dual A^*. We make A^* a Banach A-bimodule in the usual way. We call the space of all bounded derivations from A to A^* (with the operator norm) $\mathcal{D}$.
We shall look at when derivations are compact or weakly compact as linear maps.
For the majority of the talk we shall consider the convolution algebra $\ell^1(\mathbb{Z}_+)$, but first we state a general theorem.

A general theorem about (weakly-)compact derivations

Theorem (MJH thesis)

Let A be a commutative Banach algebra.

1. *If A has no non-zero, bounded, derivations of rank less than $n \in \mathbb{N}$ into A^* then it has no non-zero derivations of rank less than n into any symmetric Banach A-bimodule.*
2. *If A has no non-zero compact derivations into A^* then it has no non-zero compact derivations into any symmetric Banach A-bimodule.*
3. *If A has no non-zero weakly-compact derivations into A^* then it has no non-zero weakly-compact derivations into any symmetric Banach A-bimodule.*

Derivations from $\ell^1(\mathbb{Z}_+)$ into $\ell^1(\mathbb{Z}_+)^*$

Throughout this section we shall let A be the semigroup convolution algebra $\ell^1(\mathbb{Z}_+)$.
That is A is the Banach space $\ell^1(\mathbb{Z}_+)$ together with the product defined by

$$(ab)_n = \sum_{i+j=n} a_i b_j \qquad (a = (a_i)_{i\in\mathbb{Z}_+}, b = (b_i)_{i\in\mathbb{Z}_+}).$$

We denote the element $(0, 1, 0, 0 \ldots) \in A$ by t.

The bounded derivations from $\ell^1(\mathbb{Z}_+)$ into $\ell^1(\mathbb{Z}_+)^*$

The following is very well known and easy.

Proposition

The map

$$\Phi : \quad \mathcal{D}(A) \to \ell^\infty$$
$$D \mapsto (D(t^k)(1))_{k\in\mathbb{N}},$$

is an isometric linear isomorphism.

For $\psi \in \ell^\infty$ we set $D_\psi := \Phi^{-1}(\psi)$

The compact derivations from $\ell^1(\mathbb{Z}_+)$ into $\ell^1(\mathbb{Z}_+)^*$

The compact derivations have the following characterisation.

Theorem (MJH, PhD thesis)

Let $\psi \in \ell^\infty$. Then $D_\psi : A \to A^*$ is compact if and only if $\psi \in c_0$.

We omit the proof.

The weakly compact derivations from $\ell^1(\mathbb{Z}_+)$ into $\ell^1(\mathbb{Z}_+)^*$

The question of identifying the weakly compact derivations is much more subtle. We start with some examples.

Example (MJH, PhD thesis)

Let $\psi = (1, 1, 1 \dots)$. Then D_ψ is not weakly compact.

Not all weakly-compact derivations are compact

Proposition (Choi & MJH, 2009)

Let ψ be the indicator function of $\{2^n : n \in \mathbb{N}\} \subset \mathbb{N}$. Then D_ψ is non-compact, and the range of D_ψ is contained in $\ell^1(\mathbb{N})$.
In particular, since $\ell^1(\mathbb{N}) \subset \ell^p(\mathbb{N})$ for every $1 < p < \infty$, D_ψ factors through a reflexive Banach space and is therefore weakly compact.

Translation finiteness

We need the following definition (Ruppert, 1985).

Definition

Let $S \subseteq \mathbb{N}$. We say that S is *translation finite* (*TF* for short) if, for every sequence $n_1 < n_2 < \dots$ in $\mathbb{N}$, there exists k such that

$$\bigcap_{i=1}^{k} (S - n_k) \quad \text{is finite or empty.} \tag{1}$$

Translation finiteness is a type of "smallness". In Choi & MJH we compare it to other notions of smallness of subsets of $\mathbb{Z}_+$. For example we showed that a TF set must have Banach density 0. The following is sometimes a useful way of thinking about translation (non-)finiteness.

Lemma (Choi & MJH, 2009)

Let $S \subseteq \mathbb{N}$. Then S is non-TF if and only if there are strictly increasing sequences $(a_n), (b_n) \subset \mathbb{N}$ such that $\{a_1, \ldots, a_n\} \subseteq S - b_n$ for all n.

Our main result is the following.

Theorem (Choi & MJH, 2009)

Let $\psi \in \ell^\infty$. Then D_ψ is weakly compact if and only if, for all $\varepsilon > 0$, the set $S_\varepsilon := \{n \in \mathbb{Z}_+ : |\psi_n| > \varepsilon\}$ is TF.

Lemma (Choi & MJH, 2009)

Let $T : \ell^1(\mathbb{Z}_+) \to c_0(\mathbb{Z}_+)$ be a bounded linear map. Then the following are equivalent:

1. *T is weakly compact;*
2. *every subsequence of $(T(\delta_n))_{n\in\mathbb{N}}$ has a further subsequence which converges pointwise to some $y \in c_0(\mathbb{Z}_+)$.*

Proof of the theorem

only if : Let $\varepsilon > 0$ be such that S_ε is not TF. Let $(a_n), (b_n) \subset \mathbb{Z}_+$ be such that $a_{k-1} < a_k$, $b_{k-1} < b_k$ $(k \in \mathbb{N})$ and for each n, $b_n + \{a_0, \ldots, a_n\} \subset S_\varepsilon$. Then

$$\left| D_\psi(t^{b_n})(t^{a_m}) \right| = \frac{b_n}{b_n + a_m} \left| \psi_{b_n + a_m} \right| > \frac{b_n}{b_n + a_m} \varepsilon \quad \text{for all } m \leq n.$$

Proof of the theorem, continued

Thus, for fixed m and all large enough n, $\left|D_\psi(t^{b_n})(t^{a_m})\right| > \varepsilon$. Hence, if $(D_\psi(t^{b_n}))$ has a weakly convergent subsequence with limit ϕ then $|\phi(t^{a_m})| \geq \varepsilon$ for all m and so $\phi \notin c_0$. However $(D_\psi(t^{b_n}))_n \subset c_0$ which is weakly closed. Thus D_ψ is not weakly compact.

Proof of the theorem, continued

if: It is enough to show that if ψ is supported on a TF set then D_ψ is weakly compact.

Let $\psi \in \ell^\infty$ be supported on a TF set S. Let $(j_n) \subset \mathbb{N}$ be a strictly increasing sequence and set $(j_{0,n}) = (j_n)$. For each $i \geq 1$ we specify an integer $k_i \in \mathbb{N}$ and a sequence $(j_{i,n})_n \subset \mathbb{N}$ recursively, as follows. If there exists $k \in \mathbb{N} \setminus \{k_1, \ldots, k_{i-1}\}$ such that $j_{i-1,n} + k \in S$ for infinitely many $n \in \mathbb{N}$, let k_i be some such k.

Proof of the theorem, continued

Otherwise, let $k_i = k_{i-1}$. Let $(j_{i,n})_n$ be the enumeration of the set

$$\{j_{i-1,n} : n \in \mathbb{N}, j_{i-1,n} + k_i \in S\}$$

with $j_{i,n} < j_{i,n+1}$ for each $n \in \mathbb{N}$. Then, by induction on i, $(j_{i,n})_n$ is a subsequence of $(j_n)_n$ and, for each $l \in \{1, \ldots, i\}$, and each $n \in \mathbb{N}$, we have $j_{i,n} + k_l \in S$.

Proof of the theorem, continued

In particular, for each $i \in \mathbb{N}$, $j_{i,i} + \{k_1, \ldots, k_i\} \subset S$. Hence, by our assumption that S is TF it follows that $\{k_i : i \in \mathbb{N}\}$ is finite. Let i_0 be the smallest i for which $k_i = k_{i+1}$: then for each $k \in \mathbb{N} \setminus \{k_1, \ldots, k_{i_0}\}$, there are only finitely many $n \in \mathbb{N}$ which satisfy $j_{i_0,n} + k \in S$.

Proof of the theorem, continued.

By the Heine-Borel theorem, there exists a subsequence $(j_{n_m})_m$ of $(j_{i_0,n})_n$ such that, for each $l \in \{1, \ldots, i_0\}$, the sequence $(D_\psi(t^{j_{n_m}})(t^{k_l}))_m$ converges. Also, for every $k \in \mathbb{Z}_+ \setminus \{k_1, \ldots, k_{i_0}\}$, there exists $m(k)$ such that $D_\psi(t^{j_{n_m}})(t^k) = 0$ for all $m \geq m(k)$. Hence $D_\psi(t^{j_{n_m}})$ converges pointwise to some function supported on $\{k_1, \ldots, k_{i_0}\}$, and the result follows by the previous lemma. □

The disc algebra

Let $A(\mathbb{D})$ be the disc algebra. The situation with the disc algebra is very different.
Here the bounded derivations into the dual are exactly bounded linear maps, D, of the form

$$D(f)(g) = \int_{\mathbb{T}} f' g \, \mathrm{d}\mu.$$

In fact, by results of Morris (PhD thesis, 1993) we only need ask when these maps are bounded in the case that μ is $h\,\mathrm{d}z$ for a continuous function h on $\mathbb{T}$.

It is known, using deep results of Bourgain, that all bounded linear maps from $A(\mathbb{D})$ to its dual filter through Hilbert space (and so are weakly-compact). The proof that all bounded *derivations* are weakly compact is simpler and found in Morris (PhD thesis, 1993). We shall now give a characterisation of the bounded derivations from A to its dual.

Theorem (Choi & MJH, in preparation)

Let $AC_- = \{h \in \mathrm{Conj}\, A(\mathbb{D}) \cap AC(\mathbb{T}) : h(0) = 0\}$ *given the AC norm and let* $h \in AC_-$. *We define a derivation* $D_h : P_0(\mathbb{D}) \to A(\mathbb{D})^*$ *by*

$$D_h(f)(g) = \int_{\mathbb{T}} f'(z)g(z)h(z)\,\mathrm{d}z \qquad (f \in P_0(\mathbb{D}), g \in A(\mathbb{D})).$$

Then D_h *is bounded and* $h \mapsto D_h$ *is a bounded linear isomorphism from* AC_- *to* $\mathcal{D}(A(\mathbb{D}))$.

This gives the following immediate corollary.

Corollary

All bounded derivations from $A(\mathbb{D})$ *to its dual are compact.*

The proof of the theorem uses an argument by integration by parts taken from Morris (PhD thesis, 1993), and a result of Peller and Aleksandrov (1996) on *VMOA*.

The proof of the previous corollary also gives the following.

Proposition

Let $f \in AC_-$ and let $H_f : \mathcal{H}^2 \to L^2 \ominus \mathcal{H}^2$ be the Hankel operator with symbol f. Then $H_f(A(\mathbb{D})) \subseteq AC$ and, when considered as map from $A(\mathbb{D})$ to AC, H_f is bounded with norm equivalent to f.

Some Questions

Question

When are derivations from $\ell^1(\mathbb{Z}_+)$ to its dual p-summing?

Question

Let A be a uniform algebra. Must all bounded derivations from A to A^* be compact?

We do know that there can be a bounded, non-compact derivation from a uniform algebra A to a symmetric A-bimodule (example of Joel Feinstein, which appears in MJH's thesis).