Quantum duality and noncommutative Arens-Mackey theorem

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Introduction. We talk about the basic duality results for the quantum spaces or operator analogues of locally convex spaces. The theory is mainly due to

Ed. Effros, C. Webster and S. Winkler $(1995 - 1999)$

under the title "local operator spaces".

We are using shorter terminology

"quantum spaces" proposed by Helemskii (2004).

Instead of operator spaces - quantum normed spaces (sometimes).

The precise definitions will be done below.

In the normed case we have a well developed duality theory which is mainly due to

D. Blecher, Ed. Effros and Z.-J. Ruan (1992 – 1999).

Few attempts toward the construction of the duality theory for the general quantum spaces had been done by

C. Webster (in his Ph.D thesis 1997) - the project of the big plan.

But the basic questions remained open. Namely, on matrix bounded sets of quantum topologies compatible with the duality, Arens-Mackey type theorems, dual realization problem, matrix bornology and so on, that we want to discuss them now.

Fix a complex linear space V. The space of all $m \times n$-matrices over V is denoted by $M_{m,n}(V)$. Put $M_{n,n}(V) = M_n(V)$ and

$$M(V) = \bigcup_{n \in \mathbb{N}} M_n(V)$$

with the following (quantum) operations

$$\begin{aligned} &\oplus : M(V) \times M(V) \to M(V) \\ &v \in M_{s,t}(V), \quad w \in M_{m,n}(V), \quad v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{m+n}(V), \end{aligned}$$

and the M-bimodule operation $(M = M(\mathbb{C}))$

$$\begin{aligned} &\cdot : M \times M(V) \times M \to M(V) \\ &a = [a_{ij}] \in M_{m,s}, \quad v = [v_{ij}] \in M_{s,t}(V), \quad b = [b_{ij}] \in M_{t,n}(V), \\ &avb = \left[\sum_{k,l} a_{ik} v_{kl} b_{lj} \right] \in M_{m,n}(V). \end{aligned}$$

A finite sum like $\sum_s a_s v_s b_s$, $a_s, b_s \in M$, $v_s \in M(V)$ is called a matrix combinations.

Any linear mapping $\varphi : V \to W$ between linear spaces has the canonical linear extension

$$\varphi^{(\infty)} : M(V) \to M(W), \quad \varphi^{(\infty)}[v_{ij}] = [\varphi(v_{ij})],$$

such that

$$\varphi^{(\infty)}(M_n(V)) \subseteq M_n(W), \quad \varphi^{(\infty)}|M_n(V) = \varphi^{(n)}, \quad n \in \mathbb{N}.$$

The mapping $\varphi^{(\infty)}$ preserves the matrix operations

$$\varphi^{(\infty)}(v \oplus w) = \varphi^{(\infty)}(v) \oplus \varphi^{(\infty)}(w), \quad \varphi^{(\infty)}(avb) = a\varphi^{(\infty)}(v)\,b, \quad a, b \in M.$$

In particular, $\varphi^{(\infty)}$ preserves any matrix combination.

The matrix sets. By a matrix set $\mathfrak{B} \subseteq M(V)$ we mean a collection

$$\mathfrak{B} = (\mathfrak{b}_n) \quad \text{with} \quad \mathfrak{b}_n \subseteq M_n(V), \quad n \in \mathbb{N}.$$

We write $\mathfrak{B} \subseteq \mathfrak{M}$ iff $\mathfrak{b}_n \subseteq \mathfrak{m}_n$ for all n. A matrix set $\mathfrak{B}$ is said to be **an absolutely matrix convex set** if

$$\mathfrak{B} \oplus \mathfrak{B} \subseteq \mathfrak{B} \quad \text{and} \quad a\mathfrak{B}b \subseteq \mathfrak{B}, \quad a, b \in M, \ \|a\|, \|b\| \leq 1.$$

In particular, each $\mathfrak{b}_n$ is an absolutely convex subset in $M_n(V)$, that is, $\mathfrak{B}$ is absolutely convex. The Minkowski functional of $\mathfrak{B}$ is called *a matrix gauge.*

The Hilbert-Schmidt boundary of a matrix set.

The $\mathcal{HS}$**-boundary of a matrix set** $\mathfrak{B}$ is defined as the matrix set

$$\mathcal{HS}\left(\mathfrak{B}\right)=\left\{avb:v\in\mathfrak{B},\quad a,b\in M,\quad \left\Vert a\right\Vert_2\left\Vert b\right\Vert_2\leq 1\right\},$$

where $\left\Vert x\right\Vert_2=\operatorname{tr}\left(x^*x\right)^{1/2}$, $x\in M$. If $\mathfrak{B}$ is absolutely matrix convex then

$$\mathcal{HS}\left(\mathfrak{B}\right)\subseteq\mathfrak{B}$$

is an absolutely (not matrix) convex subset.

Quantum spaces. A filter base $\mathfrak{p} = \{\mathfrak{B}\}$ in $M(V)$ of absorbing, absolutely matrix convex sets such that $\cap\mathfrak{p} = \{0\}$ defines a (Hausdorff) polynormed (or locally convex) topology in $M(V)$. We are saying that $(V, \mathfrak{p})$ is **a quantum space with its quantum topology** $\mathfrak{p}$.

A linear mapping $\varphi : (V, \mathfrak{p}) \to (W, \mathfrak{q})$ is called **a matrix continuous** if

$$\varphi^{(\infty)} : (M(V), \mathfrak{p}) \to (M(W), \mathfrak{q})$$

is a continuous linear mapping. Note that the quantum topology $\mathfrak{p}$ in $M(V)$ inherits a polynormed topology $\mathfrak{p}|V$ in V. Conversely, each polynormed topology ς in V admits a quantization, that is,

$$\varsigma = \mathfrak{p}|V$$

for a certain quantum topology $\mathfrak{p}$ in $M(V)$.

Matrix duality. Let (V, W) be a dual pair with the duality $\langle \cdot, \cdot \rangle$, which defines *the scalar pairing*

$$M_n(V) \times M_n(W) \to \mathbb{C}, \quad \langle v, w \rangle = \sum_{i,j=1}^{n} \langle v_{ij}, w_{ij} \rangle, \quad v = [v_{ij}], w = [w_{ij}],$$

that is, $(M_n(V), M_n(W))$ is a dual pair and $\sigma(M_n(V), M_n(W)) = \sigma(V, W)^{n^2}$. *The matrix duality* is defined as

$$M(V) \times M(W) \to M, \quad \langle\langle v, w \rangle\rangle = [\langle v_{ij}, w_{st} \rangle].$$

If $\mathfrak{B} \subseteq M(V)$ is a matrix set then

$$\mathfrak{B}^{\odot} = \{w \in M(W) : \sup \|\langle\langle \mathfrak{B}, w \rangle\rangle\| \leq 1\}$$

is a matrix set in $M(W)$ called **the matrix polar of $\mathfrak{B}$**. One can also define **the classical polar**

$$\mathfrak{B}^{\circ} = (\mathfrak{b}_n^{\circ}) \subseteq M(W).$$

The weak closure $\mathfrak{B}^{-}$ is defined as the matrix set $(\mathfrak{b}_n^{-})$ of all weak closures.

Polars. Fix an absolutely matrix convex set $\mathfrak{B}$. **The Bipolar Theorem** asserts that

$$\mathfrak{B}^{\odot\odot} = \mathfrak{B}^{-} \quad \text{(Effros-Webster 1997)}$$

Using it, we derive the following

Lemma. $\mathfrak{B}^{\odot} = \mathcal{HS}\left(\mathfrak{B}\right)^{\circ}$ and $\mathcal{HS}\left(\mathfrak{B}^{\odot}\right)^{-} = \mathfrak{B}^{\circ}$.

Actually these two formulas are equivalent to the bipolar formula:

$$\mathfrak{B}^{\odot\odot} = \mathcal{HS}\left(\mathfrak{B}^{\odot}\right)^{\circ} = \left(\mathcal{HS}\left(\mathfrak{B}^{\odot}\right)^{-}\right)^{\circ} = \mathfrak{B}^{\circ\circ} = \mathfrak{B}^{-}.$$

The $\mathcal{HS}$-boundary can be useful in many other results. For instance, the Arveson-Wittstock can be reduced to Hahn-Banach:

$$\begin{aligned} \left(M\left(X\right)\cap\mathfrak{B}\right)^{\odot} &= \mathcal{HS}\left(M\left(X\right)\cap\mathfrak{B}\right)^{\circ} = \left(M\left(X\right)\cap\mathcal{HS}\left(\mathfrak{B}\right)\right)^{\circ} = \mathcal{HS}\left(\mathfrak{B}\right)^{\circ}|_{M(X)} \\ &= \mathfrak{B}^{\odot}|_{M(X)}, \end{aligned}$$

where $X \subseteq V$ is a linear subspace.

Min and Max. Let (V, W) be a dual pair and let $\mathfrak{b} \subseteq V$ be an absolutely convex set. Put $\mathfrak{b} = (\mathfrak{b}_n) \subseteq M(V)$, $\mathfrak{b}_1 = \mathfrak{b}$, $\mathfrak{b}_n = \{0\}$, $n > 1$. Define absolutely matrix convex sets

$$\mathfrak{b}_{\max} = \mathfrak{b}^{\odot\odot} \quad \text{and} \quad \mathfrak{b}_{\min} = (\mathfrak{b}^{\circ})^{\odot}$$

in $M(V)$. If $\varsigma = \{\mathfrak{b}\}$ is a neighborhood filter base of a polynormed topology in V compatible with the duality (V, W) (that is, $(V, \varsigma)' = W$), then we have the quantum topologies

$$\max \varsigma = \{\mathfrak{b}_{\max} : \mathfrak{b} \in \varsigma\} \quad \text{and} \quad \min \varsigma = \{\mathfrak{b}_{\min} : \mathfrak{b} \in \varsigma\}$$

such that

$$(\min \varsigma)\,|V = \varsigma = (\max \varsigma)\,|V.$$

Actually, if $\mathfrak{f}$ is a quantum topology in $M(V)$ such that $\mathfrak{f}|V = \varsigma$ then

$$\min \varsigma \subseteq \mathfrak{f} \subseteq \max \varsigma.$$

The weak quantum topology $\mathfrak{s}(V,W)$.

Theorem 1. Let (V,W) be a dual pair. Then

$$\min\sigma(V,W)=\max\sigma(V,W):=\mathfrak{s}(V,W),$$

that is, the weak topology admits precisely one quantization. If

$$\sigma(V,W)\subseteq\varsigma\subseteq\tau(V,W)$$

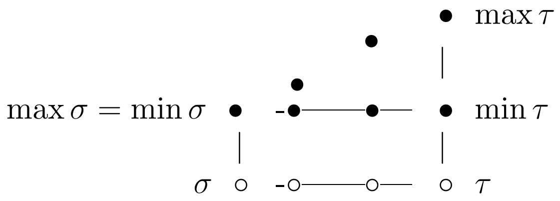
is the classical Arens-Mackey scale of all polynormed topologies compatible with (V,W) then all matrix bounded sets of the scale

$$\min\sigma(V,W)\subseteq\min\varsigma\subseteq\min\tau(V,W)$$

are the same (**noncommutative Mackey theorem**).

(For max - this result is not true!)

Thus weakly matrix bounded sets are in the middle row of the following diagram



where σ (respectively, τ) denotes the weak (respectively, Mackey) topology associated with a dual pair, black dots $\bullet$ denote the quantizations of the polynormed topologies $\circ$ between σ and τ.

The concrete model. Fix a set J.

$$\forall w \in J \quad \Longrightarrow \quad \text{"atomic" algebra } M_{n_w}.$$

$$\forall I \subseteq J \quad \Longrightarrow \quad \text{operator space } \bigoplus_{w\in I}^{\infty} M_{n_w}.$$

If $J = \{J_\kappa\}$ is a family of sets then we have the quantum space

$$\mathfrak{D}_J = \mathrm{op}\prod_{\kappa} \bigoplus_{w\in J_\kappa}^{\infty} M_{n_w} \subseteq \{\text{unbounded operators}\}$$

which is a local von Neumann algebra (Fragoulopoulou (1986), Joiţa (1999)).

$$\begin{aligned} I \quad &= \quad \{I_\alpha\} \quad \textit{is a divisor in } J = \{J_\kappa\} \\ \text{if } \forall\alpha,\ \exists!\kappa,\ I_\alpha \quad &\subseteq \quad J_\kappa, \text{ and each } J_\kappa = \cup I_\alpha \text{ for some } \alpha. \end{aligned}$$

So are the families

$$A = \{\{w\} : w \in \cup J\} \ \text{("atomic")} \quad \text{and} \quad J = \{J_\kappa\} \ \text{("top")} \quad \text{itself.}$$

Divided quantum topologies. If $I = \{I_\alpha\} \subseteq J$ is a divisor then

$$\pi_{I_\alpha}(a) = \sup\{\|a_w\| : w \in I_\alpha\}, \quad a = (a_w)_{w\in J} \in M(\mathfrak{D}_J),$$

is a matrix seminorm on $\mathfrak{D}_J = \mathrm{op}\prod_\kappa \bigoplus_{w\in J_\kappa}^{\infty} M_{n_w}$ and $\mathfrak{d}_I = \{\pi_{I_\alpha}\}$ is a quantum topology in $M(\mathfrak{D}_J)$:

$$\mathfrak{t} = \mathfrak{d}_J \quad \text{(original "top" quantum topology)},$$

$$\mathfrak{d}_I \quad \text{(divided quantum topology)},$$

$$\mathfrak{a} = \mathfrak{d}_A \quad \text{(atomic quantum topology)}.$$

Note that

$$w \in I_\alpha \subseteq J_\kappa \Rightarrow \pi_{\{w\}} \leq \pi_{I_\alpha} \leq \pi_{J_\kappa} \;\Rightarrow\; \mathfrak{a} \subseteq \mathfrak{d}_I \subseteq \mathfrak{t}.$$

If $V \subseteq \mathfrak{D}_J$ is a linear subspace then V is *a concrete quantum space with its quantum scale*

$$\mathfrak{a}|M(V) \subseteq \mathfrak{d}_I|M(V) \subseteq \mathfrak{t}|M(V).$$

Theorem 2 (Noncommutative Arens-Mackey) Let (V, W) be a dual pair. Then

$$V \subseteq \mathfrak{D}_J = \operatorname{op} \prod_{\kappa} \bigoplus_{w \in J_\kappa}^{\infty} M_{n_w}$$

is a concrete quantum space such that

$$\text{Quantizations } \{\sigma(V, W) \subseteq \varsigma \subseteq \tau(V, W)\} = \{\mathfrak{a}|M(V) \subseteq \mathfrak{d}_I|M(V) \subseteq \mathfrak{t}|M(V)\}$$

$$\mathfrak{a}|M(V) = \mathfrak{s}(V, W) \quad \text{and} \quad \mathfrak{t}|M(V) = \max \tau(V, W). \blacksquare$$

So, dividing the "top" quantum topology $\mathfrak{t}|M(V)$ one can obtain all quantum topologies compatible with the duality (V, W). Ultimately the division process is ended over the atomic level, which does not admit consecutive division, therefore $\sigma(V, W)$ admits only one quantization $\mathfrak{a}|M(V)$ (Theorem 1).

Corollary (Ruan's representation theorem). *If* $(V, \|\cdot\|)$ *is an operator space then* $V \subseteq \bigoplus_{w \in I}^{\infty} M_{n_w}$ *such that* $\pi_I = \|\cdot\|$.

Preduals. If V is a quantum space and $\mathfrak{S}$ is a matrix bornology of matrix bounded sets, then $\mathfrak{S}^{\odot} = \{\mathfrak{B}^{\odot} : \mathfrak{B} \in \mathfrak{S}\}$ is a neighborhood filter base in the quantum space $V'_{\mathfrak{S}}$ called $\mathfrak{S}$-*quantum dual of* V. If $\mathfrak{S} = \beta$ then V'_{β} is *the strong quantum dual of* V.

Consider $\mathfrak{D}_J = \operatorname{op}\prod_{\kappa} \bigoplus_{w \in J_\kappa}^{\infty} M_{n_w}$ associated with $J = \{J_\kappa\}$. We set

$$\mathcal{T}_J = \operatorname{op}\bigoplus_{\kappa} \bigoplus_{w \in J_\kappa}^{1} T_{n_w}$$

which is a quantum direct sum of the operator spaces.

Lemma. There is a canonical (topological) matrix isomorphism

$$\mathfrak{D}_J = \left(\mathcal{T}_J\right)'_{\beta},$$

Thus $$\mathfrak{a} \subseteq \mathfrak{s}\left(\mathfrak{D}_J, \mathcal{T}_J\right) \subseteq \beta\left(\mathfrak{D}_J, \mathcal{T}_J\right) = \mathfrak{t} \quad \text{in} \quad M\left(\mathfrak{D}_J\right).$$

Theorem 3 (Dual Realization). *Let V be a complete quantum space. Then*

$$V'_{\mathfrak{S}} \subseteq \mathfrak{D}_J = \mathrm{op}\prod_{\kappa} \bigoplus_{w\in J_\kappa}^{\infty} M_{n_w}$$

such that

$$\mathfrak{a}|M\left(V'\right) = \mathfrak{s}\left(\mathfrak{D}_J, \mathcal{T}_J\right)|M\left(V'\right) = \mathfrak{s}\left(V', V\right),$$

$$\mathfrak{S}^{\odot} = \mathfrak{t}|M\left(V'\right) = \beta\left(\mathfrak{D}_J, \mathcal{T}_J\right)|M\left(V'\right).$$

Corollary (Blecher-Effros). *If $(V, \|\cdot\|)$ is a complete operator space then $V^* \subseteq \bigoplus_{w\in J}^{\infty} M_{n_w}$ such that $\|\cdot\|^{\odot} = \pi_J|M\left(V^*\right)$* and

$$\mathfrak{a}|M\left(V^*\right) = \mathfrak{s}\left(\bigoplus_{w\in J}^{\infty} M_{n_w}, \bigoplus_{w\in J}^{1} T_{n_w}\right)|M\left(V^*\right) = \mathfrak{s}\left(V^*, V\right),$$

Unbounded operators. By a *quantum domain in a Hilbert space* H we mean an orthogonal family $\mathfrak{T}=\{N_\kappa\}$ of its closed subspaces such that $D=\sum_\kappa N_\kappa$ is dense in H. *The multinormed* C^**-algebra of all noncommutative continuous functions over* $\mathfrak{T}$ is defined as

$$\mathcal{C}^*_{\mathfrak{T}}(D)=\{T\in L(D):T(N_\iota)\subseteq N_\iota,\quad T|N_\iota\in\mathcal{B}(N_\iota)\}$$

which consists of closable unbounded operators on H. So, $\mathcal{C}^*_{\mathcal{E}}(D)$ is a quantum space $(p_\iota(a)=\|a|N_\iota^n\|$, $a\in M_n(\mathcal{C}^*_{\mathfrak{T}}(D)))$. Note that

$$(\mathfrak{D}_J,\mathfrak{t})=\operatorname{op}\prod_\kappa\overset{\infty}{\bigoplus_{w\in J_\kappa}}M_{n_w}\subseteq\mathcal{C}^*_{\mathfrak{T}}(D)\ \text{ (quantum space inclusion)}$$

with $N_\kappa=\bigoplus\limits_{w\in J_\kappa}\mathbb{C}^{n_w}$ and $H=\bigoplus\limits_{w\in\cup J_\kappa}\mathbb{C}^{n_w}$. Further,

$$(\mathfrak{D}_J,\mathfrak{a})=\operatorname{op}\prod_\kappa\prod_{w\in J_\kappa}M_{n_w}\subseteq\mathcal{C}^*_{\mathfrak{A}}(D)\ \text{ (quantum space inclusion)}$$

with $\mathfrak{A}=\{\mathbb{C}^{n_w}\}$ an "atomic" quantum domain.

Quantum domains. The quantum domain D can be quantized as

$$D_q = \mathrm{op}\sum_{\kappa} N_{\kappa,q},$$

where q is a quantization over a class of normed spaces.

Theorem 4. If $\mathfrak{T} = \{N_\kappa\}$ is a quantum domain then $D_q = \left((D_q)'_\beta\right)'_\beta$ and

$$\mathcal{C}^*_{\mathfrak{T}}(D) \subseteq \mathcal{MC}(D_c)_\beta \text{ (quantum space inclusion).}$$

If $\mathfrak{A} = \{\mathbb{C}^{n_w}\}$ and $\mathfrak{A}' = \{\mathbb{C}^{m_v}\}$ are atomic quantum domains with the same union space $D = \sum_w \mathbb{C}^{n_w} = \sum_v \mathbb{C}^{m_v}$ then

$$D_{q_1} = \mathrm{op}\sum_{w} \left(\mathbb{C}^{n_w}\right)_{q_1} = \mathrm{op}\sum_{v} \left(\mathbb{C}^{m_v}\right)_{q_2} = D_{q_2} = \max D$$

for all quantizations q_1 and q_2. In particular,

$$\mathcal{C}^*_{\mathfrak{A}}(D) \subseteq \mathcal{MC}(\max D)_\beta \text{ (quantum space inclusion).}$$

Locally compact and locally trace class operators. Let $\mathfrak{T}=\{N_\kappa\}$ be a quantum domain with the union space $D=\sum N_\kappa$. We set

$$\begin{aligned}\mathcal{K}_{\mathfrak{T}}^{*}\left(D\right) &= \left\{T\in\mathcal{C}_{\mathfrak{T}}^{*}\left(D\right):T|N_\kappa\in\mathcal{K}\left(N_\kappa\right)\right\}\subseteq\mathcal{C}_{\mathfrak{T}}^{*}\left(D\right)\\ &\quad\text{(locally compact operators)}\\ \mathcal{T}_{\mathfrak{T}}^{*}\left(D\right) &= \operatorname{op}\sum_{\kappa}\mathcal{T}\left(N_\kappa\right)\subseteq\mathcal{C}_{\mathfrak{T}}^{*}\left(D\right)\\ &\quad\text{(locally trace class operators).}\end{aligned}$$

Theorem 5. $\mathcal{T}_{\mathfrak{T}}^{*}\left(D\right)_{\beta}^{\prime}=\mathcal{C}_{\mathfrak{T}}^{*}\left(D\right)$ and $\mathcal{K}_{\mathfrak{T}}^{*}\left(D\right)_{\beta}^{\prime}=\mathcal{T}_{\mathfrak{T}}^{*}\left(D\right)$. In particular, $\mathfrak{D}_J=\left(\mathcal{T}_J\right)_{\beta}^{\prime}$ and $\mathcal{K}_{\mathfrak{T}}^{*}\left(D\right)\subseteq\mathcal{C}_{\mathfrak{T}}^{*}\left(D\right)$ is just the inclusion

$$\mathcal{K}_{\mathfrak{T}}^{*}\left(D\right)\hookrightarrow\left(\mathcal{K}_{\mathfrak{T}}^{*}\left(D\right)_{\beta}^{\prime}\right)_{\beta}^{\prime}.$$

Matrix bornology. A bornology $\mathfrak{N}$ in $M(V)$ of matrix sets such that $\operatorname{amc}(\mathfrak{B}) \in \mathfrak{N}$ whenever $\mathfrak{B} \in \mathfrak{N}$, is called *a matrix bornology.*

If (V, W) is a dual pair and $\mathfrak{S}$ is a quantum topology in $M(V)$ compatible with the duality then $\mathfrak{S}^{\odot}$ is a matrix bornology in $M(W)$. Consider a quantum subspace $X \subseteq V$ with its quantum topology $M(X) \cap \mathfrak{S}$, and the canonical duality (X, Y) with $Y = W/X^{\perp}$, associated with (V, W). We have a quotient mapping $\varphi : W \to Y$.

Theorem 6. $\varphi : (W, \mathfrak{S}^{\odot}) \to \left(Y, (M(X) \cap \mathfrak{S})^{\odot}\right)$ is an exact matrix quotient mapping of quantum bornological spaces, that is, $\varphi^{(\infty)}(\mathfrak{S}^{\odot}) = (M(X) \cap \mathfrak{S})^{\odot}$.

Corollary. If $X \subseteq V$ is an operator space inclusion then

$$\varphi^{(\infty)}(\operatorname{ball} M(V^*)) = \operatorname{ball} M(X^*),$$

where $\varphi : V^* \to X^*$, $\varphi(f) = f|_X$. In particular, $V^*/X^{\perp} = X^*$ up to the matrix isometry.

Let (V, W) be a dual pair and let $\mathfrak{S}$ be a matrix bornology in $M(V)$ of weakly bounded matrix sets. Then $\mathfrak{S}^{\odot}$ defines a quantum topology in $M(W)$. Put $M(W_{\mathfrak{S}}) = (M(W), \mathfrak{S})$.

Theorem 7. *Let $\varphi : (T, D) \to (V, W)$ be a weakly continuous linear mapping of the dual pairs such that*

$$\varphi' : (W, V) \to (D, T) \text{ is a weak}^{*}\text{isomorphism onto its range}$$

and

$$(\varphi')^{(\infty)} : M(W_{\mathfrak{S}}) \to M(D_{\mathfrak{T}}) \text{ is a matrix isomorphism onto its range,}$$

then

$$\varphi : (T, \mathfrak{T}) \to (D, \mathfrak{S})$$

is a matrix quotient mapping of the quantum bornological spaces, that is,

$$\varphi^{(\infty)}(\mathfrak{T})^{-} = \mathfrak{S}^{-}.$$

Corollary. *Let V be a complete quantum space and let $\mathfrak{S}$ be a matrix bornology in $M(V)$ of $\sigma(V, V')$-bounded matrix sets. There is a matrix quotient mapping*

$$\varphi : (\mathcal{T}_J, \beta) \to (V, \mathfrak{S})$$

of the matrix bornological spaces, where β is a matrix bornology of all matrix bounded sets in $M(\mathcal{T}_J)$.

In the normed case Corollary is reduced to the fact that each complete operator space is a matrix quotient of an L^1-direct sum of finite dimensional trace class algebras up to a matrix isometry (Blecher 1992).

Application. The main technical machinery that we have used is the inductive limits of operator spaces. The inductive limits of operator spaces have been successfully used in the quantum moment problems too. The quantum operator valued measures are treated as matrix contractive and matrix positive linear mappings between certain inductive limits of operator spaces, see

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