Cohomology of strong semilattices of Banach algebras

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Based on (slight simplifications of)

[C1] *Simplicial homology and Hochschild cohomology of Banach semilattice algebras*, Glasgow Math. J., 2006

[C2] *Simplicial homology and cohomology of strong semilattices of Banach algebras*, Houston J. Math. (*to appear* 2009/10) cf. `math.FA/0609450`

Examples of semilattices

A *semilattice* is a commutative semigroup where each element is idempotent.

For each semilattice L, there is a canonical partial order $\preceq$ on L, defined by

$$e \preceq f \iff e = ef (= fe) .$$

Note that if $x, y, z \in L$ and $z \preceq x$, $z \preceq y$, then $z \preceq xy$. Thus xy is the greatest lower bound ('meet') of x and y in $(L, \preceq)$.

Conversely, if $(P, \preceq)$ is a partially ordered set where each pair of elements has a g.l.b., we may define an associative product on P by taking $xy := x \wedge y$. The resulting semigroup $P_\wedge$ is a semilattice.

Examples.

- $\{0, 1, \ldots, n\}$ *with the usual (partial) order.*
- $\mathbb{N}$ *with the usual (partial) order.*
- $\mathbb{N}_\vee$, *i.e.* $\mathbb{N}$ *with the reverse (partial) order.*
- $\{\theta, 1, \ldots, n\}$ *with* $j \cdot k = \begin{cases} j & \text{if } j = k \\ \theta & \text{otherwise.} \end{cases}$
- $\{\theta\} \sqcup \mathbb{N}$ – *as above.*

ℓ^1-gradings over semilattices

Let $\mathfrak{A}$ be a Banach algebra. Suppose that the underlying Banach space of $\mathfrak{A}$ decomposes as an ℓ^1-direct sum

$$\mathfrak{A} = \ell^1\text{-}\bigoplus_{\alpha\in L} A_\alpha$$

where the indexing set L is a semilattice, and that moreover:

- each A_α is a closed subalgebra of $\mathfrak{A}$;
- $A_\alpha \cdot A_\beta \subseteq A_{\alpha\beta}$ for all $\alpha, \beta \in L$.

Then we say $\mathfrak{A}$ is *graded over L*, and refer to A_α as the *fibre over α*.

Examples.

- *A semidirect product*

$$\begin{tikzcd} A_0 \arrow[r] & \mathfrak{A} \arrow[r, shift left, "q"] & A_1 \arrow[l, shift left, "\rho"] \end{tikzcd}$$

- *An ℓ^1-direct sum*

$$\mathfrak{A} = \ell^1\text{-}\bigoplus_{i\in\mathbb{I}} A_i$$

The structure of L gives us some information about how elements in different fibres of $\mathfrak{A}$ multiply together; however, it only tells us in **which fibre** the product of two given elements lies, and not **where in the fibre.**

Strong ℓ^1-gradings over semilattices

Let L be a semilattice; let $(A_\alpha)_{\alpha\in L}$ be a family of Banach algebras, indexed by L.

Suppose that for every $e \succeq f$ in L, we have a contractive algebra homomorphism $\phi_{f,e} : A_e \longrightarrow A_f$, and suppose that the family $(\phi_{f,e})$ satisfies the following compatibility conditions:

- $\phi_{e,e}$ is the identity homomorphism on A_e, for each $e \in L$;
- if $e \succeq f \succeq g$ in L then $\phi_{g,f} \circ \phi_{f,e} = \phi_{g,e}$.

We can use these data to construct a Banach algebra $\mathfrak{A}$ that is graded over L. The underlying Banach space is $\ell^1\text{-}\bigoplus_{\alpha\in L} A_\alpha$, while the multiplication is defined by taking

$$x \cdot y := \phi_{ef,e}(x)\phi_{ef,f}(y) \qquad \text{for } x \in A_e,\ y \in A_f,$$

and extending by linearity and continuity.

We say that $\mathfrak{A}$ is *strongly graded over L*. In analogy with terminology from semigroup theory, we also refer to $\mathfrak{A}$ as a *strong semilattice of Banach algebras*.

Examples. *The ℓ^1-direct sum of the previous slide is strongly graded. The semidirect product example is in general* **not** *strongly graded: if it is, then the semidirect product is isomorphic to a direct sum of algebras.*

Examples constructed from semigroups

Given a semigroup S, we may equip $\ell^1(S)$ with convolution multiplication: this is defined on basis elements of ℓ^1 by

$$e_s \cdot e_t := e_{st} \qquad (s, t \in S),$$

and then extended by linearity and continuity.

Remark. *If S is a semigroup, graded over a semilattice L, then the convolution algebra $\ell^1(S)$ will be ℓ^1-graded over L. Furthermore: if the grading of S is a strong grading (i.e. is given by a system of compatible homomorphisms between fibres), then the grading of $\ell^1(S)$ over L will also be a strong grading.*

Let $\mathbb{G}$ be a semigroup. We say $\mathbb{G}$ is a *Clifford semigroup* if

- each $a \in \mathbb{G}$ has a *relative inverse*, that is, there exist $e, b \in \mathbb{G}$ with $ea = a = ae$ and $ab = e = ba$;
- all idempotents in $\mathbb{G}$ commute with each other.

Theorem. [Clifford, 1941] *Let $\mathbb{G}$ be a Clifford semigroup, L the set of its idempotents. Then $\mathbb{G}$ is strongly graded over L, and for each $x \in L$ the fibre G_x is a group.*

By the remarks above, $\ell^1(\mathbb{G}) = \ell^1\text{-}\bigoplus_{x\in L} \ell^1(G_x)$ is a strong semilattice of ℓ^1-group algebras.

A topological variant

Let G be a locally compact, abelian group. The measure algebra $\mathrm{M}(G)$ is a rather large and complicated object.

Independent work of J. Inoue and J. L. Taylor (1970s) considered a certain closed, unital subalgebra of $\mathrm{M}(G)$, called the *spine* of the measure algebra and denoted by $\mathrm{M}_*(G)$.

It may be defined as the closed linear span of all maximal subalgebras of $\mathrm{M}(G)$ which are isometrically algebra-isomorphic to some L^1-group algebra.

Although this is not obvious from the definition, $\mathrm{M}_*(G)$ has a natural, strong ℓ^1-grading over a certain structure semilattice L. Each fibre is isometrically isomorphic to $L^1(G_\tau)$ where G_τ denotes the completion of G with respect to a certain group topology τ.

Thus $\mathrm{M}_*(G) = \ell^1\text{-}\bigoplus_{\tau\in L} L^1(G_\tau)$ is a strong semilattice of abelian L^1-group algebras, and could be regarded as the convolution algebra of a certain kind of locally compact abelian Clifford semigroup.

Hochschild cohomology (1)

Fix a Banach algebra A and a Banach A-bimodule M. Let

$$\mathcal{C}^n(A, M) = \{\text{bounded } n\text{-linear maps } A \times \ldots \times A \to M\}$$

(the space of *continuous n-cochains*)

We form the *Hochschild cochain complex*

$$0 \to \mathcal{C}^0(A, M) \xrightarrow{\delta^0} \mathcal{C}^1(A, M) \xrightarrow{\delta^1} \mathcal{C}^2(A, M) \xrightarrow{\delta^2} \ldots$$

where δ^n is the Hochschild *coboundary operator* in degree n.

Elements of $\operatorname{Ker} \delta^n$ are *n-cocycles*; elements of $\operatorname{Im} \delta^{n-1}$ are *n-coboundaries*; and the vector space of n-cocycles modulo n-coboundaries, which we denote by $\mathcal{H}^n(A, M)$, is the *nth Hochschild cohomology group* of A with coefficients in M.

In general the cohomology groups are complete, **seminormed** spaces.

Notation. *When $M = A^*$, regarded as a dual A-bimodule in the usual way, we shall abbreviate $\mathcal{C}^\bullet(A, A^*)$ to $\mathcal{C}^\bullet(A)$, $\mathcal{Z}^\bullet(A, A^*)$ to $\mathcal{Z}^\bullet(A)$ and so forth.*

Hochschild cohomology (2)

Example.

- 1-*cocycles are the same as continuous derivations;*
- 1-*coboundaries are the same as inner derivations;*
- *hence* $\mathcal{H}^1(A, M)$ *is the space of continuous derivations modulo inner ones.*

A Banach algebra A is *amenable* if and only if $\mathcal{H}^1(A, M^*) = 0$ for **every** Banach A-bimodule M. If this is the case, then by using a "dimension-shift" argument, it follows that $\mathcal{H}^n(A, M^*) = 0$ for any Banach A-bimodule M.

Taking $M = A$, we see that every amenable Banach algebra A satisfies

$$\mathcal{H}^n(A) = 0 \quad \text{for all } n \geq 1. \tag{1}$$

More generally, a Banach algebra A which satisfies (1) is said to be *simplicially trivial*.

Theorem 1. [C., 2006] *Let* L *be a semilattice. Then* $\ell^1(L)$ *is simplicially trivial.*

Note that, in general, $\ell^1(L)$ is not amenable. In fact, it is amenable if and only if L is finite (Duncan & Namioka, 1978).

A decomposition theorem

Given $\alpha \in L$, there is for each n an obvious 'restriction map' from $\mathcal{C}^n(\mathfrak{A})$ to $\mathcal{C}^n(A_\alpha)$. Taking the direct product (i.e. the ℓ^∞ sum) over all $\alpha \in L$, we obtain a restriction map

$$\mathrm{rest}_n : \mathcal{C}^n(\mathfrak{A}) \longrightarrow \prod_{\alpha \in L} \mathcal{C}^n(A_\alpha)$$

Denote the right-hand side by $\mathcal{C}^n_{\mathrm{diag}}(\mathfrak{A})$.

Because A_α is a subalgebra of $\mathfrak{A}$, the sequence (rest_n) forms a chain map between chain complexes, and hence induces maps

$$\rho_n : \mathcal{H}^n(\mathfrak{A}) \longrightarrow \mathcal{H}^n_{\mathrm{diag}}(\mathfrak{A}) := H^n \left[\prod_{\alpha \in L} \mathcal{C}^\bullet(A_\alpha) \right]$$

Theorem 2. [C.] *ρ_n is an isomorphism of seminormed spaces, for every n.*

Corollary 3. *If the family $(A_\alpha)_{\alpha \in L}$ is 'uniformly biflat', then $\mathfrak{A}$ is simplicially trivial.*

This occurs, for instance, if each A_α has the form $L^1(G_\alpha)$ for some locally compact, amenable group G_α.

A warning example

How important is this choice of coefficient module? For instance, what can we say about $\mathcal{H}^n(\mathfrak{A}, \mathfrak{A})$?

It turns out that even in degree 1, we might not get good decomposition results. The following example is taken from work of Bowling & Duncan (1998).

Theorem 4. *Let L be a semilattice consisting of infinitely many pairwise orthogonal idempotents; let S_3 denote the symmetric group of order 6; and put $\mathbb{G} = L \times S_3$. Then $\mathcal{H}^1(\ell^1(\mathbb{G}), \ell^1(\mathbb{G})) \neq 0$.*

(Note that in this case, $\ell^1(\mathbb{G})$ is even **biflat**.)

We do at least have the following result:

Theorem 5. [ibid.] *Let $\mathbb{G}$ be a Clifford semigroup which has only finitely many idempotents. Then $\mathcal{H}^1(\ell^1(\mathbb{G}), \ell^1(\mathbb{G})) = 0$.*

Remark. *The case of a* **single** *idempotent is the statement that, for a discrete group G, every derivation from $\ell^1(G)$ to $\ell^1(G)$ is inner; this is an old result of Johnson & Ringrose* (1969)*.*

Why do we need a 'strong grading'?

When calculating the Hochschild cohomology of a Banach algebra $\mathfrak{A}$, there is a well-established arsenal of 'averaging techniques' that one can try to deploy. These work particularly well if one has an **amenable** algebra $\mathfrak{C}$ that acts on $\mathfrak{A}$, and which is 'large enough' to detect some of the structure in $\mathfrak{A}$.

Suppose $\mathfrak{A} = \ell^1\text{-}\bigoplus_{\alpha\in L} A_\alpha$ is strongly graded over L. If each fibre A_α were unital, then we would obtain an isometric algebra embedding of $\ell^1(L)$ into (the centre of) $\mathfrak{A}$.

Such an embedding need not exist, in general. However, we do at least have a central action of $\ell^1(L)$ on $\mathfrak{A}$, defined as follows: given $x, y \in L$ and $a \in A_y$, we put

$$x \cdot a = a \cdot x := \phi_{xy,y}(a) .$$

For algebras which are graded over a semilattice, but not strongly graded, it is not clear if we can construct a useful analogue of this $\ell^1(L)$-action. Further progress can be made for the ℓ^1-convolution algebras of certain semigroups, where a grading – but not a strong grading – exists. (Joint work with Gourdeau & White, *in preparation*.)

Normalized cochains

If we are trying to prove that a given cocycle is a coboundary, a natural first step is to replace it with an equivalent cocycle which has additional ‘nice’ structure.

Let K be a unital Banach algebra that acts on another Banach algebra A. A cochain $\psi \in \mathcal{C}^n(A, M)$ is said to be *K-normalized*, or *K-multimodular*, if for every $a_1, \ldots, a_n \in A$ and every $c \in K$,

$$\begin{aligned} \psi(ca_1, a_2, \ldots, a_n) &= c\psi(a_1, a_2, \ldots, a_n) \\ \psi(\ldots, a_i c, a_{i+1}, \ldots) &= \psi(\ldots, a_i, ca_{i+1}, \ldots) \\ & \qquad\qquad (1 \leq i \leq n-1) \\ \psi(a_1, \ldots, a_n c) &= \psi(a_1, \ldots, a_n)c \end{aligned}$$

For each n let $\mathcal{C}^n_K(A, M)$ denote the subspace of K-normalized n-cochains. It turns out that these form a subcomplex, so we can define $\mathcal{Z}^n_K(A, M)$, $\mathcal{B}^n_K(A, M)$ and hence the *nth K-relative cohomology group* $\mathcal{H}^n_K(A, M)$.

When K is **amenable**, standard techniques ensure that every cochain from A to a **dual** bimodule M is equivalent to a K-normalized cochain.

[More precisely, we can construct a chain projection ν from $\mathcal{C}^\bullet(A, M)$ onto $\mathcal{C}^\bullet_K(A, M)$ such that id $- \nu$ is **null-homotopic**.]

"$\ell^1(L)$-normalised $\equiv$ diagonal"

For sake of legibility, we henceforth write $\mathcal{C}^n_L$ instead of $\mathcal{C}^n_{\ell^1(L)}$. Similarly, we shall use the notation $\mathcal{Z}^n_L$ and $\mathcal{H}^n_L$.

If M is an $\mathfrak{A}$-bimodule, then in general we don't seem to have a good description of $\mathcal{C}^n_L(\mathfrak{A}, M^*)$. However, on taking $M = \mathfrak{A}$ we can do better: in this case, the composite map

$$\mathcal{C}^\bullet_L(\mathfrak{A}) \hookrightarrow \mathcal{C}^\bullet(\mathfrak{A}) \xrightarrow{\mathrm{rest}^\bullet} \mathcal{C}^\bullet_{\mathrm{diag}}(\mathfrak{A})$$

turns out to be an isomorphism of chain complexes. Using this, we can deduce Theorem 2 from the following technical result.

Theorem 6. **[C.,** *ibid*.**]** *Define $\mu^n : \mathcal{C}^n(\mathfrak{A}) \to \mathcal{C}^n(\mathfrak{A})$ by*

$$\mu^n\psi(a_1, \ldots, a_n)(a_0) = \psi(p \cdot a_1, \ldots, p \cdot a_n)(p \cdot a_0)$$

where ψ is a simplicial n-cochain, $a_i \in A_{x(i)}$ for $0 \le i \le n$ and $p = x(0) \cdot x(1) \cdots x(n)$. Then:

(i) *$\mu^\bullet$ is a chain projection onto the subcomplex $\mathcal{C}^\bullet_L(\mathfrak{A})$*

(ii) *$\mu^\bullet$ is chain homotopic to the identity map.*

In particular, $\mathcal{H}^\bullet_L(\mathfrak{A}) \cong \mathcal{H}^\bullet(\mathfrak{A})$.

The case of trivial fibres...

As part of work in progress, we can simplify some of the proof techniques in **[C1]** and **[C2]**. We outline the main ideas below, for the special case $\mathfrak{A} = \ell^1(L)$.

In the following discussion we abbreviate $\mathcal{C}^n(\ell^1(L))$ to $\mathcal{C}^n(L)$, and similarly for cocycles and cohomology.

It is natural and tempting to argue along the following lines. Let $\psi \in \mathcal{Z}^n(L)$. We wish to find $\varphi \in \mathcal{C}^{n-1}(L)$ such that $\delta\varphi = \psi$, and we know this is possible when L is finite.

So, given $x_0, x_1, \ldots, x_n \in L$, let $\langle \mathbf{x} \rangle$ denote the **finite** sub-semilattice generated by $x_0, \ldots, x_n$. We know there exists $\varphi_{\langle \mathbf{x} \rangle} \in \mathcal{C}^{n-1}(\langle \mathbf{x} \rangle)$ such that

$$\delta\varphi_{\langle \mathbf{x} \rangle}(x_1, \ldots, x_n)(x_0) = \psi(x_1, \ldots, x_n)(x_0)$$

and so we might try defining $\varphi \in \mathcal{C}^{n-1}(L)$ by

$$\varphi(y_1, \ldots, y_{n-1})(y_0) := \varphi_{\langle \mathbf{y} \rangle}(y_1, \ldots, y_{n-1})(y_0)$$

However, it is not clear if $\delta\varphi$ equals ψ. Some correction term seems to be needed.

. . . c'est remaudit

Given $x_0, \ldots, x_m \in L$, if we restrict attention to $\mathcal{C}^\bullet(\langle \mathbf{x} \rangle)$, we can use **amenability** of $\ell^1(\langle \mathbf{x} \rangle)$ to obtain maps

$$\mathcal{C}^{n-1}(\langle \mathbf{x} \rangle) \xleftarrow{s^{\mathbf{x}}} \mathcal{C}^{n}(\langle \mathbf{x} \rangle) \xleftarrow{s^{\mathbf{x}}} \mathcal{C}^{n+1}(\langle \mathbf{x} \rangle)$$

such that $\delta s^{\mathbf{x}} + s^{\mathbf{x}} \delta$ is the identity map. In particular, if $\psi \in \mathcal{Z}^n(L)$ then $\delta s^{\mathbf{x}} \psi$ and ψ agree when restricted to $\langle \mathbf{x} \rangle$.

Now, suppose we can find a bounded linear map $\sigma_{n-1} : \mathcal{C}^n(L) \to \mathcal{C}^{n-1}(L)$ which satisfies

$$\delta \sigma_{n-1} \delta = \delta$$

and has the following 'locality' property: for any $(x_0, \ldots, x_n) \in L^{n+1}$, if ψ vanishes when all entries are restricted to $\ell^1(\langle \mathbf{x} \rangle)$, then so does $\delta \sigma_{n-1} \psi$.

We can then use σ_{n-1} and maps of the form $s^{\mathbf{x}}$, to construct a bounded linear map $\sigma_n : \mathcal{C}^n(L) \to \mathcal{C}^{n+1}(L)$ which satisfies

$$\delta \sigma_n + \sigma_n \delta = \text{identity map}$$

and which also has the 'locality' property.

Since $\ell^1(L)$ is commutative, taking $\sigma_0 = 0$ works, and by induction we obtain σ_1, σ_2, . . . as desired.

Cohomology with symmetric coefficients

If A is a commutative Banach algebra, then under certain (strong) side conditions one can use knowledge of its simplicial cohomology groups to calculate cohomology groups with **symmetric** coefficients.

Proposition 7. *Let A be a unital, commutative Banach algebra whose underlying Banach space has the form $L^1(\Omega)$ for some measure space Ω. Suppose that $\mathcal{H}^n(A, A^*) = 0$ for all $n \geq 1$. Then*

$$\mathcal{H}^n(A, M^*) = 0$$

for all $n \geq 1$ and any **symmetric** *Banach A-bimodule M.*

If G is a locally compact, abelian group, then $\mathrm{M}_*(G)$ satisfies the conditions of this proposition, so we have the following corollary.

Corollary 8. *Let N be a symmetric, dual Banach $\mathrm{M}_*(G)$-bimodule. Then $\mathcal{H}^n(\mathrm{M}_*(G), N) = 0$ for all $n \geq 1$.*

Remark. *We can relax the hypotheses on the bimodule N slightly; the corollary remains true if the bimodule N is symmetric and complemented (as a Banach space) in its second dual. This is the case for $N = \mathrm{M}_*(G)$ for instance.*

Epilogue: the spine of a Fourier-Stieltjes algebra

Introduced and studied by Ilie & Spronk (2007).

Let G be a locally compact group, $\mathrm{A}(G)$ its *Fourier algebra*, and $\mathrm{B}(G)$ its *Fourier-Stieltjes algebra*.

$$\boxed{\text{For } G \text{ abelian, } \mathrm{A}(G) \cong L^1(\widehat{G}) \text{ and } \mathrm{B}(G) \cong \mathrm{M}(\widehat{G}).}$$

If $\eta : G \longrightarrow H$ is a continuous group homomorphism with dense range, we get an isometric algebra homomorphism $\eta^* : \mathrm{A}(H) \longrightarrow \mathrm{B}(G)$. The *spine* of $\mathrm{B}(G)$, denoted by $\mathrm{A}^*(G)$, is the closed subalgebra of $\mathrm{B}(G)$ generated by the ranges of all such homomorphisms η^*. This generalizes the definitions of Inoue and Taylor that were mentioned earlier.

Just as in the abelian setting, $\mathrm{A}^*(G)$ admits a strong ℓ^1-grading:

$$\mathrm{A}^*(G) = \ell^1\text{-}\bigoplus_{\tau \in L} \mathrm{A}(G_\tau)$$

where L is a certain semilattice of group topologies on G, and G_τ is the completion of G w.r.t. the topology τ.

Can we get a "completely bounded version" of Corollary 8 (at least, within a reasonably large and natural class of groups)?