$C(K)$ representations of real Banach Algebras

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Representations via the state space

$\mathcal{A}$ will denote a commutative real Banach algebra with unit e, where $\|e\| = 1$.

THEOREM Suppose that whenever $a, b \in \mathcal{A}$,

$$a^2 + b^2 = e \text{ implies } \|a^2\| \leq 1. \tag{1}$$

Then, $\mathcal{A}$ is isomorphic to a $\mathcal{C}(K)$-space.

Sets considered,

$$S = \{\varphi \in \mathcal{A}^* : \|\varphi\| = \varphi(e) = 1\}; \quad \mathcal{A}_+ = \{a^2 : a \in \mathcal{A}\}$$

We represent $\mathcal{A}$ on S by $a \to \hat{a}$ where $\hat{a}(\varphi) = \varphi(a)$.

LEMMA Suppose $\mathcal{A}$ satisfies condition (1). Let $\varphi \in \mathcal{A}^*$ with $\|\varphi\| = 1$. Then, $\varphi \in S$ if and only if $\varphi \geq 0$ on $\mathcal{A}_+$.

Proof

Suppose φ in S. If $\|a^2\| < 1$ then by (1), $\|e - a^2\| \leq 1$ so that $\varphi(e - a^2) \leq 1$ and thus $\varphi(a^2) \geq 0$.

Suppose $\varphi \geq 0$ on $\mathcal{A}_+$. If $\|u\| < 1$ then $e - u$ is in $\mathcal{A}_+$ so that $\varphi(e - u) \geq 0$. It follows that $\varphi(e) = \|\varphi\|$.

LEMMA Under (1), if $e-b \notin \mathcal{A}_+$ then there is $\varphi \in S$ such that $\varphi(b) > 1$.

Proof

If $\|u\| < 1$ then $e-u$ is in $\mathcal{A}_+$. Hahn- Banach gives ψ in $\mathcal{A}^*$, with $\|\psi\| = 1$ satisfying

$$\psi(e-b) > t\psi(e-u) \text{ for all } u \text{ in } \mathcal{A},\ \|u\| < 1, \text{ and all } t > 0\ .$$

Now put $\varphi = -\psi$.

CONCLUSION If $a \in \mathcal{A}_+$ and $\|a\| > 1$ then $\|\hat{a}\|_{\infty,S} \geq 1$ (because $e - a \notin \mathcal{A}_+$) so that

$$\|a\| = \|\hat{a}\|_{\infty,S} \text{ on } \mathcal{A}_+.$$

Also

If $\|\hat{b}\|_{\infty,S} < 1$ then, by lemma, $e \pm b \in \mathcal{A}_+$ so that

$$2\|b\| \leq \|e+b\| + \|e-b\| = \|1+\hat{b}\|_{\infty,S} + \|1-\hat{b}\|_{\infty,S} \leq 4.$$

Thus $\|\cdot\|$ and $\|\hat{\cdot}\|_{\infty,S}$ are equivalent.

Restrict the functions $\hat{a}$ to $X = \overline{\mathrm{ext}S}$.

Claim: $a \to \hat{a}$ becomes multiplicative.

Let $\|a^2\| < 1$ and let ψ be in ext$S$. Let $\psi_c(b) = \psi(cb)$. Then

$$\psi = \psi_{a^2} + \psi_{e-a^2}$$

so that $\psi = \psi_{a^2}$ or $\psi = \psi_{e-a^2}$. It follows that $\psi(a^2 b) = \psi(a^2)\psi(b)$ for all b. Since $\mathcal{A} = \mathcal{A}_+ - \mathcal{A}_+$, the claim follows. Stone-Weierstrass does the rest.

EXAMPLE 1 $\mathcal{A} = C(X)$ with $\|f\| = \|f_+\|_{\infty,X} + \|f_-\|_{\infty,X}$ satisfies the condition in the theorem so the isomorphism need not be an isometry.

The condition

$$\|a\|^2 \leq \|a^2 + b^2\|,$$

which implies (1), gives isometry.

Representations via the maximal ideal space

The *complex spectrum* of a in $\mathcal{A}$ is the set

$$\sigma_{\mathbf{C}}(a) = \{\, (s,t) \in \mathbf{R}^2 : (a-s)^2 + t^2 \text{ not invertible in } \mathcal{A}\}.$$

PROPOSITION For a in $\mathcal{A}$, the complex spectrum $\sigma_{\mathbf{C}}(a)$ is nonempty and

$$\sup_{(s,t)\in\sigma_{\mathbf{C}}(a)} (s^2+t^2)^{1/2} = r(a).$$

Proof (sketch) Let $\varphi \in \mathcal{A}^*$ and put

$$u(s,t) = \varphi((a-s)((a-s)^2+t^2)^{-1}),$$

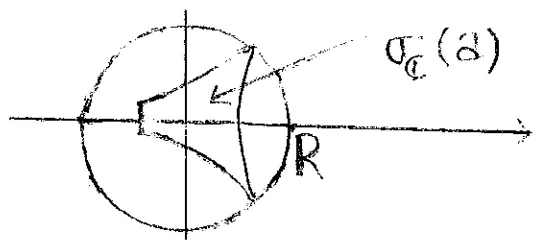
a harmonic function.

Thus, for $\|a^2 - 2sa\| < (s^2+t^2)$,

$$\|(a-s)((a-s)^2+t^2)^{-1}\| = \|(a-s)(a^2-2sa+s^2+t^2)^{-1}\|$$
$$\leq \frac{\|a-s\|}{s^2+t^2} \cdot \sum_{n=0}^{\infty} \left\| \frac{a^2-2sa}{s^2+t^2} \right\|^n = \frac{\|a-s\|}{s^2+t^2} \cdot \frac{1}{1-\frac{\|a^2-2sa\|}{s^2+t^2}},$$

which tends to 0 as $(s^2+t^2) \to \infty$.

If u were everywhere defined then, by the maximum modulus principle, $u \equiv 0$. which is not true.



The harmonic conjugate of u is $v(s,t) = \varphi(t((a-s)^2+t^2)^{-1})$ so that

$$w = u + iv \text{ is analytic outside } \sigma_{\mathbf{C}}(a), \quad w(z) = \sum_0^{\infty} a_n\, 1/z^n, \quad |z| > R.$$

Then

$$w(s) = \sum_0^{\infty} a_n\, 1/s^n = \varphi(\,(a-s)^{-1}) = -\sum_0^{\infty} \varphi(a^n)\, 1/s^n.$$

The series have same radius of convergence, R. Uniform boundedness gives $\|a^n/s^n\|$ bounded. Thus $r(a) \leq R$. Since the first series is convergent for $|z| > R$, we have $r(a) \geq R$.

Homomorphisma on $\mathcal{A}$ over R

$\mathcal{M}$ maximal ideal$\Rightarrow$ $\mathcal{A}/\mathcal{M}$ is a division algebra $\Rightarrow$ $\mathcal{A}/\mathcal{M}$ isomorphic to **R** or **C**.

X is the space of homomorphisms with the Gelfand topology. We represent $\mathcal{A}$ on X as usual . We want conditions giving *real-valued* functions.

LEMMA $(s,t) \in \sigma_{\mathbf{C}}(a)$ if and only if $\varphi(a) = s + it$ for some homomorphism φ.

So we want conditions where the complex spectrum is real.

$$\begin{aligned} \|a\|^2 &\le k_1 \|a^2 + b^2\|, \quad (a, b \in \mathcal{A}), \quad (1) \\ \|a^2\| &\le k_2 \|a^2 + b^2\|, \quad (a, b \in \mathcal{A}), \quad (2) \\ r(a^2) &\le k_3\, r(a^2 + b^2), \quad (a, b \in \mathcal{A}), \quad (3) \end{aligned}$$

LEMMA (1) and (2) are equivalent and imply that $\|\cdot\|$ and $r(\cdot)$ are equivalent norms on $\mathcal{A}$.

Proof Clearly (1) implies (2) (with $k_2 = k_1$). If (2) holds:

$$\begin{aligned} \|a\| = 1/4\|(e+a)^2 - (e-a)^2\| &\le 1/4(\|(e+a)^2\| + \|(e-a)^2\|) \\ &\le k_2/4(\|(e+a)^2 + (e-a)^2\| + \|(e-a)^2 + (e+a)^2\|) \\ &= k_2 \|e + a^2\| \le k_2 (1 + \|a^2\|). \end{aligned}$$

Replace a by ta: $a^2 = 0 \Rightarrow a = 0$. Thus $\|a^2\| \le 1$ implies $\|a\|^2 \le 4k_2^2$ and it follows that $\|a\|^2 \le 4k_2^2\|a^2\|$ for all a. Conclusion, (1) holds with $k_1 = 4k_2^3$.

The inequality $\|a\|^2 \le 4k_2^2\|a^2\|$ and induction leads to

$$\|a\|^{2^n} \le (4k^2)^{2^n} \|a^{2^n}\|.$$

Taking roots and letting n tend to infinity we obtain,

$$\|a\| \le 4k^2 r(a).$$

Since $r(a) \le \|a\|$ we are done.

$$\|a\|^2 \leq k_1 \|a^2 + b^2\|, \quad (a, b \in \mathcal{A}), \quad (1)$$

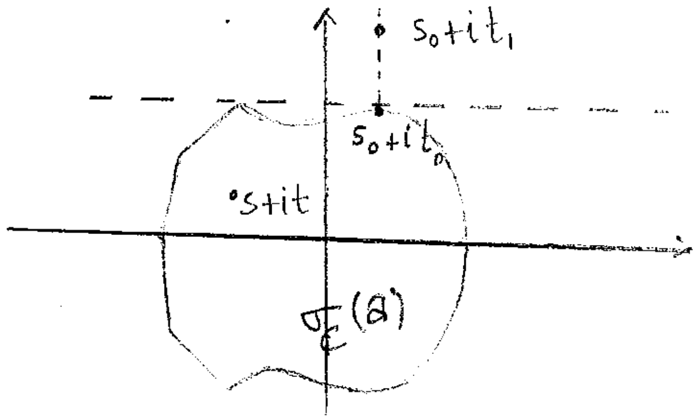
$$\|a^2\| \leq k_2 \|a^2 + b^2\|, \quad (a, b \in \mathcal{A}), \quad (2)$$

$$r(a^2) \leq k_3 \, r(a^2 + b^2), \quad (a, b \in \mathcal{A}), \quad (3)$$

THEOREM If (3) holds, then $\sigma_{\mathbf{C}}(a)$ is a subset of $\mathbf{R}$ and if (1) or (2) then, $\mathcal{A}$ is isomorphic to a $\mathcal{C}(K)$-space.

The proof depends on the following lemma:

LEMMA Suppose $\sigma_{\mathbf{C}}(a)$ is a *not* a subset of $\mathbf{R}$. Then there is $F \subset X$ and b, u in $\mathcal{A}$ such that $b = 1$ on F, $|b| < 1$ on $X \setminus F$ and $1 + u^2 = 0$ on F.



Proof Let b satisfy

$$b^{-1} = (a - s_0)^2 + t_1^2.$$

For x in X, write $\hat{a}(x) = s + it$ and $\hat{a}(x_0) = s_0 + it_0$. Then

$$|\hat{b}(x)| = |((s + it - s_0)^2 + t_1^2)^{-1}| \leq (t_1^2 - t^2)^{-1} \leq (t_1^2 - t_0^2)^{-1} = \hat{b}(x_0),$$

with equality if and only if $\hat{a}(x) = s_0$. Now put

$$F = \{\, x \,:\, \hat{b}(x) = \hat{b}(x_0)\} \quad \text{and} \quad u = t_0^{-1}(a - s_0).$$

and normalize b

$$\|a\|^2 \le k_1 \, \|a^2 + b^2\|, \quad (a, b \in \mathcal{A}), \quad (1)$$

$$\|a^2\| \le k_2 \, \|a^2 + b^2\|, \quad (a, b \in \mathcal{A}), \quad (2)$$

$$r(a^2) \le k_3 \, r(a^2 + b^2), \quad (a, b \in \mathcal{A}), \quad (3)$$

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Proof of theorem If $\sigma_{\mathbf{C}}(a)$ is a *not* a subset of $\mathbf{R}$, let F, b and u be as in Lemma. From (3) we get

$$r(b^{2n}) \le k \, r(b^{2n} + u^2 b^{2n}) = k \, r(b^{2n}(1 + u^2)).$$

The left hand side equals 1 and the right hand side tends to 0 as n increases.

If (1) or (2) hold, the spectral radius semi-norm is equivalent to the norm on $\mathcal{A}$. Since

$$r(a) = \sup_{(s,t) \in \sigma_{\mathbf{C}}(a)} (s^2 + t^2)^{1/2} = \sup_{x \in X} |\hat{a}(x)|,$$

the Stone-Weierstrass theorem shows that the map $a \to \hat{a}$ from $\mathcal{A}$ into $C(X)$ is an isomorphism.

REMARK If (1) holds with $k_1 = 1$ then $\|a^2\| = \|a\|^2$ so that $r(a) = \|a\|$. In this case we have isometry.

EXAMPLE 2 The algebra $\mathcal{A} = C^{(1)}[0,1]$,

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$

is a commutative real Banach algebra with unit. Clearly, if $t \neq 0$, $(f-s)^2+t^2$ is invertible, so that (s,t), for $t \neq 0$, is not in the complex spectrum of f, and $\sigma(f) = f([0,1])$ for every f in $\mathcal{A}$. It follows that the Gelfand transform in this case is just the inclusion map from $\mathcal{A}$ into $C[0,1]$. Thus $\mathcal{A}$ satisfies (3) with $k = 1$.

EXAMPLE 3 Let $\mathcal{A}$ be the subalgebra of $\ell_1(\mathbf{Z})$, consisting of those sequences $(a_k)_{k\in\mathbf{Z}}$ for which $a_k = a_{-k}$ for every natural number k,with convolution as multiplication and the norm given by

$$\|(a_k)_{k\in\mathbf{Z}}\| = \sum_{\mathbf{Z}} |a_k|.$$

The space of homomorphisms can be identified with the unit circle.

Both are examples of real Banach algebras which consist of hermetian elements in *-algebras. In this situation every a in $\mathcal{A}$ satisfying $\hat{a} > 0$ is a square because we have complex functional calculus at our disposal. However, there are elements a in $\mathcal{A}$ satisfying $\hat{a} \geq 0$ which are not squares. This is a consequense of Katznelson's square root theorem which says that if that were the case, $\mathcal{A} = C(X)$.