Ideals and structure in operator algebras

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Abstract

We begin by describing some recent ideas in operator space theory, such as the noncommutative topology of operator algebras. We connect some of this to some ideas in Banach algebra theory, and use it to study the structure of a possibly new class of nonselfadjoint algebras.

Operator algebra $=$ closed subalgebra of $B(H)$, H Hilbert

unital: has an identity of norm 1

approximately unital: has a cai

Think: 'noncomm. function algebra/uniform algebra'

or think: 'partial C^*-algebra'

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- We often use C^*-algebras generated the algebra A
- or the diagonal $\Delta(A) = A \cap A^*$, a C^*-algebra
- Projection in A $=$ norm 1 (orthogonal) idempotent in A

BA2007: Title: A nice class of Banach algebras

From the BA2007 talk:

In view of the developments in the subject of operator algebras (from an operator space perspective) in the last decade or so, it seems to be a good time to reexamine the connections

$$\text{Banach algebras} \quad \leftrightarrow \quad \text{operator algebras}$$

For example, operator algebras are a 'halfway house' between (noninvolutive) Banach algebras and C^*-algebras

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For example, operator algebras are a 'halfway house' between (noninvolutive) Banach algebras and C^*-algebras

- It is interesting to ask questions inspired by Banach algebra theory, about operator algebras

- Conversely, it is interesting to ask about operator algebra properties which might extend to (certain) classes of Banach algebras

We begin by reviewing some recent theory:

I. One-sided ideals, hereditary subalgebras, noncommutative peak sets, etc

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- If (e_t) is the left cai of an r-ideal J, then $e_t \to e$ weak* in A^{**}, where e is a left identity for $J^{\perp\perp}$

- $J^{\perp\perp} = eA^{**}$

- e is the support projection of J

- This is all straightforward. Now something deep happens:

Theorem (Hay, B-H-Neal) The support projections of r-ideals in an operator algebra A are exactly the projections $e \in A^{**}$ such that $e \in (eA^{**}e \cap A)^{\perp\perp}$.

We say these projections in A^{**} are open w.r.t. A. The 'perp' is a closed projection

- These are exactly the projections in A^{**} which are open in the C^*-algebraic sense of Akemann, with respect to any C^*-algebra containing A.

- Thus we enter the realm of the noncommutative topology of C^*-algebras, a deep subject.

- E.g. $\exists$ noncommutative Urysohn lemma (Akemann), which we exploited to create noncommutative peak interpolation for operator algebras

- Get many simple sounding but deep results, e.g.

Theorem $\exists$ bijective correspondence between right ideals with left cai, and the left ideals with right cai ('ℓ-ideals')

This is surprising, since it probably is not true for Banach algebras

The correspondence with peaking: Let A be a unital subalgebra of $B = C(K)$

The peak sets for A in K, correspond to the open projections which are the support projection of an r-ideal of the form $\overline{(1-x)A}$, for $x \in \mathrm{Ball}(A)$

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This is a Theorem (Glicksberg, ...) ...

in our language, this theorem says generic r-ideals of A are closures of unions of r-ideals of the form $\overline{(1-x)A}$, for $x \in \mathrm{Ball}(A)$

Whether this theorem holds in the noncommutative setting is probably the biggest open question in the noncommutative peaking theory/theory of r-ideals. We have many reformulations and consequences

Eg. It is related to the important question of whether every r-ideal J is proximinal (that is, $d(x, J)$ is achieved for all $x \in A$)

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HSA's have several equivalent definitions in the C^*-algebra setting:

e.g. 1) given a right ideal J of a C^*-algebra B, J^* is a left ideal, and $D = J \cap J^*$ constitute precisely the hereditary subalgebras

2) In terms of the second dual, $D = pB^{**}p \cap B$ for an open projection p

3) they are the $*$-subalgebras D with $DBD \subset D$

4) many other characterizations

- In [B-H-Neal] we began to develope the theory of HSA's in (nonselfadjoint) operator algebras

- Can define a HSA to be an approximately unital subalgebra D of A such that $DAD \subset A$

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Just as in C^* case:

Theorem (BHN) For a projection $p \in A^{**}$, the following are equivalent:

(1.) p is open in A^{**}
(2.) p is open in B^{**} for any C^*-alg B containing A
(3.) p is the support projection of an r-ideal of A
(4.) p is the support projection of an ℓ-ideal of A
(5.) p is the support proj. of an HSA of A
(6.) p is lower-semicontinuous on the state space $S(A)$
(7.) The set $\{\phi \in S(A) : \phi(p) = 0\}$ is a weak*-closed face of $S(A)$.

- HSA's are just the subalgebras $A \cap pA^{**}p$, for open projection p

- HSA's are also just the intersection of an r-ideal with its canonically associated left ideal

- The ideal/HSA structure is linked to what is going on in a containing C^*-algebra B, eg. $B = C_e^*(A)$

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- These results seem to have nice applications, for example solved a 15 yr old problem in the theory of operator modules

- We rely on a deep NC variant of the theory of 'peak sets' in function algebra theory (Hay)

- $\exists$ variant for any classes of B. algebras? Most of it makes sense ...

• Summary so far: for an operator algebra A, the following are all essentially the same theory:

ℓ-ideals, r-ideals, HSA's, the noncommutative topology of A (i.e. theory of open and closed projections with respect to A), noncommutative peak and p-sets, noncommutative peak interpolation

and requires adapting deep techniques from C^*-algebra theory (eg. the noncommutative Urysohn lemma) to get the best results

We now turn to very recent work, a project with my students Almus and Sharma

Disclaimers:

- We are almost exclusively concerned with the isometric theory (as opposed to isomorphic)

- There may be some things in the Banach algebra literature we have missed

- May have missed some tricks

- We have tons of open questions

Def: annihilator C^*-algebra $= c_0$-direct sum of algebras of the form $\mathbb{K}(H)$

Many, many characterizations of these in the literature: E.g.

Theorem Let A be a C^*-algebra. The following are equivalent:

(i) There is a faithful $*$-representation $\pi : A \to \mathbb{K}(H)$ for some Hilbert space H.

(ii) A is $*$-isomorphic to $\oplus_i^\circ \mathbb{K}(H_i)$ for Hilbert spaces H_i.

(iii) The spectrum of every commutative $*$-subalgebra of A is discrete topologically.

(iv) A is an ideal in its bidual.

(v) For every $x \in A$, $\mathrm{Sp}_A(x) \setminus \{0\}$ is discrete topologically.

(vi) Every closed right ideal J in A is of the form eA for a projection in $M(A)$.

(vii) The sum of all minimal right ideals in A is dense in A.

(viii) A is Kaplansky-dual/annihilator/modular annihilator/compact/...

- It is interesting to ask, which operator algebras have the Banach algebraic properties in the last theorem: annihilator algebras, Kaplansky-dual algebras, modular annihilator algebras, compact Banach algebras, algebras with dense socle, isomorphic to a subalgebra of the compacts $\mathbb{K}(H)$, etc.

- There do not seem to be simple answers ... can't say for example that two of the above properties are equivalent.

This was one of several initial motivation for this project

Aside: we later will give some answers. For example we will get a theorem that if any one of the above Banach algebraic properties holds, plus a reasonable extra property, then get a (different but nice) conclusion

Proposition Simplicity and semiprimeness passes to HSA's

Question: Semisimplicity passes to HSA's?

Proposition If A has a right cai, and if p is a projection in A^{**} such that $pA \subset A$, then $Ap \subset A$

Proposition If $\Delta(A)$ acts nondegenerately on A, then $M(\Delta(A)) = \Delta(M(A))$

We define a notion based on the noncommutative topology of operator algebras (from Section I). We will say that an operator algebra A is nc-discrete if it satisfies the equivalent conditions in the next result:

Proposition For an approximately unital operator algebra A the following are equivalent:

(i) Every open projection e in A^{**} is also closed (in the sense that $1 - e$ is open).

(ii) The open projections in A^{**} are exactly the projections in $M(A)$.

(iii) Every r-ideal J of A is of the form eA for a projection $e \in M(A)$.

(iv) Every ℓ-ideal J of A is of the form Ae for a projection $e \in M(A)$.

(v) Every HSA of A is of the form eAe for a projection $e \in M(A)$.

If any of these hold then $\Delta(A)$ is an annihilator C^*-algebra

Example: If A is an ℓ-ideal in its bidual, then A is nc-discrete, and indeed every projection in A^{**} is open

- Say A is Δ-*dual* if $\Delta(A)$ is an annihilator C^*-algebra, and $\Delta(A)$ acts nondegenerately on A

- nc-discrete implies Δ-dual under reasonable conditions, but not vice-versa

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- Nc-discrete algebras are reminiscent of Kaplansky's dual algebras, in that the 'left (resp. right) annihilator' operation is a lattice anti-isomorphism between the lattices of one-sided M-ideals of A

  Indeed, $R(L(J)) = J$ (resp. $L(R(J)) = J$) for an r-ideal (resp. ℓ-ideal) J in A

- Also, e.g. for nc-discrete algebras, the intersection, and the closure of the sum of, of any family of r-ideals (resp. ℓ-ideals) in A is again a r-ideal (resp. ℓ-ideal)

III. Existence of r-ideals:

Write 1 for the identity in A^1 if A is a nonunital algebra.

Proposition If A is an approximately unital operator algebra, which is an ideal in an operator algebra B, then $\overline{(1-x)A}$ is an r-ideal in A for all $x \in \text{Ball}(B)$

A right ideal J of A is regular if there exists $y \in A$ such that $(1-y)A \subset J$.

We say that J is 1-regular if this can be done with $\|y\| \leq 1$.

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Proposition If the major open problem mentioned in Section I has an affirmative solution, then r-ideals are always 1-regular.

- In a unital function algebra, r-ideals are always 1-regular. But this is probably deep (uses a theorem of Glicksberg?).

- There are interesting connections here to the topics of proximinality, and noncommutative peak interpolation

- Say that an ideal of A is trivial if it is (0) or A

Theorem If A is a Banach algebra, and $x \in \mathrm{Ball}(A)$, then

(1) If A is unital then $1 - x$ is left invertible in A iff $1 - x$ is right invertible in A.

If A is nonunital then x is left quasi-invertible in A iff x is right quasi-invertible in A.

(2) Every 1-regular right ideal of A is trivial iff the spectral radius $r(a) < \|a\|$ for all $a \in A \setminus \mathbb{C}1$; and iff $\mathrm{Ball}(A) \setminus \{1\}$ is composed entirely of quasi-invertible elements.

In the literature?

Can get more for operator algebras (Banach algebras?)

Theorem Let A be an operator algebra, and $x \in \mathrm{Ball}(A)$

(1) The following are equivalent (and are equivalent to the other conditions in the last theorem):

(i) $1 - x$ is invertible in A.

(ii) $\|1 + x\| < 2$.

(iii) 1 is not an eigenvalue of x.

(2) Every 1-regular right ideal of A is trivial iff $\|1+x\| < 2$ for all $x \in \mathrm{Ball}(A)$ with $x \neq 1$, and iff $\nu(x) < \|x\|$ for all $x \in A \setminus \mathbb{C}1$. Here ν is the numerical radius in A^1.

• For nc-discrete algebras the existence of nontrivial r-ideals and 1-regular ideals, is also equivalent to the existence of projections in $M(A)$ and A

Question: For a nc-discrete algebra A, do A and $M(A)$ have nontrivial projections simultaneously

• For nc-discrete algebras the existence of nontrivial r-ideals and 1-regular ideals, is also equivalent to the existence of projections in $M(A)$ and A

Question: For a nc-discrete algebra A, do A and $M(A)$ have nontrivial projections simultaneously

Theorem Let A be an operator algebra which contains nontrivial 1-regular ideals (or equivalently, $\mathrm{Ball}(A) \setminus \{1\}$ is not composed entirely of quasi-invertible elements). Then proper maximal r-ideals of A exist.

Indeed, if $y \in \mathrm{Ball}(A)$ is not quasi-invertible then $(1-y)A$ is contained in a proper (regular) maximal r-ideal.

The unit ball of the intersection of the 1-regular maximal r-ideals of A is composed entirely of quasi-invertible elements of A.

IV. Matricial algebras

We say that an operator algebra A is 1-matricial if it has a set of matrix units $\{T_{ij}\}$, whose span is dense in A, and the norm of $q_k = T_{kk}$ is always 1.

Matrix units: $T_{ij}T_{kl} = \delta_{jk}T_{il}$

We almost exclusively focus on 1-matricial algebras, and only care about the separable case (although nonseparable case is identical)

We prefer the following equivalent description:

Consider a (finite or infinite) sequence $T_1, T_2, \cdots$ of invertible operators on a Hilbert space K, with $T_1 = I$

Set $H = \ell^2 \otimes^2 K = K^{(\infty)} = K \oplus^2 K \oplus^2 \cdots$

Define $T_{ij} = E_{ij} \otimes T_i^{-1} T_j \in B(H)$, and let A be the closure of the span of the T_{ij}

A σ-matricial algebra is a c_0-direct sum of 1-matricial algebras

(Not a good name, but less of a mouthful that σ-1-matricial, etc)

Proposition 1-matricial algebras A are approximately unital, topologically simple, and are primitive, compact, modular annihilator algebras, with dense socle

A (first) Wedderburn theorem: If A is a topologically simple operator algebra with a sequence of nonzero mutually orthogonal algebraically minimal projections (q_k) with $\sum_k q_k = 1$ strictly, then A is unitarily isomorphic to a 1-matricial algebra.

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- There are other Wedderburn-type theorems for operator algebras in the literature, for example due to Helemskii, which are no doubt related to the ones we obtain

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Proposition A 1-matricial algebra A is a right (resp. left, two-sided) ideal in its bidual iff $q_1 A$ (resp. Aq_1, $q_1 A$ and Aq_1) is reflexive

Lemma A separable 1-matricial algebra A is completely isomorphic to $\mathbb{K}(\ell^2)$ iff A is isomorphic to a C^*-algebra as Banach algebras, and iff $(\|T_k\|\|T_k^{-1}\|)$ is bounded

Conjecture: A 1-matricial algebra A is bicontinuously isomorphic to $\mathbb{K}(\ell^2)$ iff it is an annihilator algebra

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Conjecture: A 1-matricial algebra A is bicontinuously isomorphic to $\mathbb{K}(\ell^2)$ iff it is an annihilator algebra

• If the K in the definition of a 1-matricial algebra A is finite dimensional, we say that A is subcompact

Lemma If A is a subcompact 1-matricial algebra, then A is completely isometrically isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$

Conversely, if a 1-matricial algebra A is bicontinuously isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$, then A is bicontinuously isomorphic to a subcompact 1-matricial algebra

(These are ideals in their bidual)

Example Let $K = \ell^2$, and $T_k = E_{kk} + \frac{1}{k} I$.

- In this case $q_1 A$ is not reflexive, indeed $q_1 A \cong c_0$
- On the other hand, $A q_1$ is a column Hilbert space
- By an earlier result, A is a left ideal in its bidual, but is not a right ideal in its bidual

This is interesting since any C^*-algebra which is a left ideal in its bidual is also a right ideal in its bidual

Building exotic 1-matricial algebras: If $T_2, \cdots$ are arbitrary elements in a unital operator space X, then there exists a 1-matricial algebra A with $q_1 A$ bicontinuously isomorphic to the closure of the span of $E_{1k} \otimes T_k$

- So you can make $q_1 A$ pretty much anything you want
- One can simultaneously make Aq_1 as nice as one wants

- For example, can take X to be Pisiers operator Hilbert space, or your favourite exotic operator space

Proposition σ-matricial algebras are Δ-dual

- In fact one pleasant feature of 1-matricial algebras is that one can get a grip on their second duals:

Lemma If A is a 1-matricial algebra then A^{**} is the collection of infinite matrices $[\beta_{ij} T_i^{-1} T_j]$, for scalars β_{ij}, which are bounded operators on $K^{(\infty)}$

Corollary Let A be a σ-matricial algebra. Any projection $p \in A^{**}$ lies in $M(A)$ and is open. Hence A is nc-discrete. Also,

$$\Delta(A^{**}) = \Delta(A)^{**} = M(\Delta(A)) = \Delta(M(A)).$$

For the next results we need some definitions concerning projections p in A

Say p is $*$-minimal if it has no nontrivial subprojection

Say p is algebraically minimal if pAp is one dimensional

A second Wedderburn-type theorem: Let A be an approximately unital semiprime operator algebra. The following are equivalent:

(i) A is completely isometrically isomorphic to a σ-matricial algebra.

(ii) A is the closure of $\sum_k q_k A$ for mutually orthogonal algebraically minimal projections $q_k \in A$.

(iii) A is the closure of the joint span of minimal right ideals qA, for algebraically minimal projections $q \in A$.

(iv) A is Δ-dual, and every $*$-minimal projection in A is algebraically minimal.

(v) A is Δ-dual, and every nonzero projection in A dominates a nonzero algebraically minimal projection in A.

(vi) A is nc-discrete, and every projection p in $M(A)$ with the property that $pAp \setminus \mathbb{C}p$ contains no nonzero selfadjoint elements in A, is algebraically minimal in A.

(etc)

- If A is a one-sided ideal in A^{**}, then can drop some of these hypotheses

• Connections with modular annihilator algebras/compact algebras/dense socle/etc:

Theorem Let A be a semiprime operator algebra such that $\Delta(A)$ acts nondegenerately on A. Suppose also that every $*$-minimal projection in A also cannot properly dominate a nonzero idempotent in A. TFAE:

(i) A is completely isometrically isomorphic to a σ-matricial algebra.

(ii) A is compact.

(iii) A is a modular annihilator algebra.

(iv) The socle of A is dense.

(v) A is semisimple and the spectrum of every element in A has no nonzero limit point.

(vi) Every orthogonal projection in A is finite rank (or equivalently, in the socle) and $\Delta(A)$ an annihilator C^*-algebra

• Can rephrase this in other ways...

- Any semiprime operator algebra contains a canonical σ-matricial algebra:

For an operator algebra A, define the h-socle to be the HSA with support projection f equal to the 'join' of all the algebraically minimal projections

Theorem Let A be a semiprime operator algebra. Then the h-socle of A is a σ-matricial algebra. It is the biggest HSA in A

Conversely, every HSA in a (σ-)matricial algebra is a (σ-)matricial algebra

- If A is a σ-matricial algebra, then the r-ideals, ℓ-ideals, and HSA's of A are of a very nice form

For example, the r-ideals are just an orthogonal direct sum of right ideals of form eA, for algebraically minimal projections e

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Question What other Banach algebraic properties do 1-matricial algebras have?

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Question What other Banach algebraic properties do 1-matricial algebras have?

Another question Are 1-matricial algebras bicontinuously isomorphic iff they are completely isomorphic?

Lots more questions...

V. More characterizations of annihilator C^*-algebas

Theorem Any semiprime approximately unital operator algebra A with dense socle, such that every minimal right ideal of A equals pA for a projection $p \in A$, is completely isometrically isomorphic to a C^*-algebra of compact operators

Corollary Any semisimple approximately unital operator algebra such that every closed left ideal is contractively A-complemented, is completely isometrically isomorphic to a C^*-algebra of compact operators.

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Corollary Any semisimple approximately unital operator algebra such that every closed left ideal is contractively A-complemented, is completely isometrically isomorphic to a C^*-algebra of compact operators.

Question: If an approximately unital operator algebra has the property that all closed right ideals have a left cai, is a C^*-algebra?

- There exists a theory of complemented Banach algebras, but it uses quite different techniques, and does not get the above results (and vice versa).

- There is a sticky point when trying to connect the theories that would be nice to resolve

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- We imagine that the isomorphic version of the last result is true:

A semisimple approximately unital operator algebra such that every closed left ideal in A is A-complemented, is isomorphic to a C^*-algebra of compact operators?

- Cant quite finish the proof ... I guess theres still a lot we do not understand

A final characterization of annihilator C^*-algebras, in terms of some notions of Hamana:

Theorem If A is a C^*-algebra, the following are equivalent:

(i) A is an annihilator C^*-algebra.

(ii) A^{**} is an essential extension of A.

(iii) A^{**} is an injective envelope of A.

(iv) $I(A^{**})$ is an injective envelope of A.

(v) Every surjective complete isometry $T : A^{**} \longrightarrow A^{**}$ maps A onto A.

(vi) A is nuclear and A^{**} is a rigid extension of A.

Reference: *Ideals and structure of operator algebras,* Almus, Blecher, and Sharma