

# The James–Schreier Space

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# Definitions

In this talk:  $x = (\alpha_j) \subset \mathbb{R}^{\mathbb{N}}$ ;

$$\nu_p(x, A) := \left( \sum_{n \in A} |\alpha_n|^p \right)^{1/p}$$

Schreier norm:  $\|x\|_{S_p} := \sup\{\nu_p(x, A) : A \subseteq \mathbb{N} \text{ admissible}\}$

$$\mu_p(x, A) := \left( \sum_{j=1}^k |\alpha_{n_j} - \alpha_{n_{j+1}}|^p \right)^{1/p}$$

James norm:  $\|x\|_{J_p} := \sup\{\mu_p(x, A) : A \subseteq \mathbb{N}\}$

James–Schreier norm:  $\|x\|_{V_p} := \sup\{\mu_p(x, A) : A \subseteq \mathbb{N} \text{ permissible}\}$

# Definitions

Two ways of defining  $J_p$ :

## Definition 1

Let  $J_p$  be completion of  $c_{00}$  with respect to the  $J_p$  norm.

## Definition 2

Let  $J_p$  be those sequences in  $c_0$  which have finite  $J_p$  norm.

These two definitions describe the same space as

$$\|(I - P_n)x\|_{J_p} \rightarrow 0 \Leftrightarrow \|x\|_{J_p} < \infty,$$

that is, the unit vectors  $(e_i)$  are a Schauder basis of  $J_p$  in Definition 2.

# Definitions

## Definition - $V_p$

Let  $V_p$  be completion of  $c_{00}$  with respect to the  $\|\cdot\|_{V_p}$ -norm.

## Definition - $W_p$

Let  $W_p$  be those sequences in  $c_0$  which have finite  $\|\cdot\|_{V_p}$ -norm.

Here, the spaces  $V_p$  and  $W_p$  are not equal or isomorphic.

## Equivalent Definition - $V_p$

Let  $V_p$  to be the closure of the the linear span of the basic sequence  $(e_i)$  in  $W_p$  with respect to the  $\|\cdot\|_{V_p}$ -norm.

These two definitions of  $V_p$  are trivially equivalent. However, unlike the James space, now the basic sequence  $(e_i)$ , the basis by definition, of  $V_p$ , does not span  $W_p$ .

# Definitions

Similarly for the Schreier space:

## Definition - $Z_p$

Let  $Z_p$  be those sequences in  $c_0$  which have finite  $\|\cdot\|_{S_p}$ -norm.

## Definition - $S_p$

Let  $S_p$  to be the closure of the the linear span of the basic sequence  $(e_i)$  in  $Z_p$  with respect to the  $S_p$  norm.

**e.g.**  $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$  has finite  $\|\cdot\|_{S_1}$ -norm, so is in  $Z_1$ , but is not in  $S_1$ .

**Claim:**  $Z_p$  is the second dual of the Schreier space  $S_p$ .

# Shrinking basis

Let  $(b_n)$  be a basis for a Banach space  $X$ .

## Definition and Theorem

Let  $(f_n)$  be the sequence of **coordinate functionals** for  $(b_n)$ , that is,  $f_n : \sum \alpha_m b_m \mapsto \alpha_n$ ; then each  $f_n$  is bounded.

## Definition

A basis  $(b_n)$  is **shrinking** if and only if the coordinate functionals  $(f_n)$  are a basis for  $X^*$ .

## Proposition

For  $p > 1$  the unit vectors  $(e_i)$  are a shrinking basis for  $V_p$ .

# Second Dual

- Define the natural map  $\kappa : V_p \rightarrow V_p^{**}$  by  $\kappa(x) : f_i \mapsto x_i$ .
- The shrinking basis for  $V_p$  allows us to construct an isometric isomorphism from  $V_p^{**}$  to

$$X_{V_p} := \left\{ (\alpha_n) \subseteq \mathbb{C}^{\mathbb{N}} : \sup_m \left\| \sum_{n=1}^m \alpha_n e_n \right\|_{V_p} < +\infty \right\} \cong W_p \oplus \mathbb{C}1.$$

# The James–Schreier Banach Algebra

For  $1 \leq p < \infty$  the space  $X_{V_p}$  is a commutative  $*$ -algebra under the norm  $\|\cdot\|_{V_p}$  equipped with pointwise multiplication and pointwise complex conjugation as involution, and has separately continuous product.

The Banach space  $X_{V_p}$  is a commutative Banach  $*$ -algebra with identity  $e_0 = (1, 1, \dots)$  under the  $\|\cdot\|_{V_p}$ -norm equipped with pointwise multiplication.

$W_p$  is a  $*$ -subalgebra of  $X_{V_p}$ .

$V_p$  is a  $*$ -ideal of  $W_p$  and  $X_{V_p}$ .

# The James–Schreier Banach Algebra

We define  $\chi_n$  to be  $\sum_{i=1}^n e_i = (1, 1, \dots, 1, 0, \dots)$ . The sequence  $(\chi_n)$  is a bounded approximate identity of projections of  $\|\cdot\|_{V_p}$ -norm 1 in the Banach  $*$ -algebra  $V_p$ , and they are contained in  $c_{00}$ .

The commutative Banach  $*$ -algebra  $V_p$  is weakly amenable, but not amenable.

$V_p$  is Arens regular. Hence, the multiplier algebra of  $V_p$ ,  $(V_p^{**}, \square)$ ,  $X_{V_p}$ ,  $W_p \oplus \mathbb{C}1$  are all isometrically isomorphic.

# James' Theorem

## James' Theorem - 1950

A Banach space with an unconditional basis, is reflexive if and only if it has no embedded copies of  $c_0$  or  $l_1$ .

## Corollary

The James space has no copies of  $c_0$  or  $l_1$ , but is not reflexive. So  $J_p$  has no unconditional basis.

But the James–Schreier space,  $V_p$  *does* contain copies of  $c_0$ —it is  $c_0$ -saturated! So a different approach is necessary.

# Pełczyński's Property (u)

## Pełczyński's Property (u) - 1958

A Banach space  $X$  has **Pełczyński's property (u)** if for all weak Cauchy sequences  $(x_n) \subset X$  there exists a sequence  $(y_n) \subset X$  such that for all  $f \in X^*$

$$\left\langle x_n - \sum_{i=1}^n y_i, f \right\rangle \rightarrow 0 \text{ as } n \rightarrow \infty$$

and  $\sum_{n=1}^{\infty} |\langle y_n, f \rangle|$  is convergent.

# Pełczyński's Property (u)

**Reminder:**  $X$  has Pełczyński's property (u) if for all weak Cauchy sequences  $(x_n) \subset X$  there exists a weakly unconditionally convergent (WUC) series  $\sum_{n=1}^{\infty} y_n$  such that  $(x_n - \sum_{i=1}^n y_i)_{n \in \mathbb{N}}$  is weakly null.

Every subspace of a space with Pełczyński's property (u) also has Pełczyński's property (u).

Every Banach space with an unconditional basis has property (u). In particular  $c_0$  and  $S_p$  have it.

To show a Banach space does not have an unconditional basis it is enough to show it doesn't have property (u).

# Pełczyński's Property (u) - James Space

## Theorem - Bessaga and Pełczyński

Every weak unconditionally convergent (WUC) series in a Banach space  $X$  is unconditionally convergent if and only if  $X$  contains no copy of  $c_0$ .

**Proposition:**  $J_p$  does not have Pełczyński's property (u).

**Proof (by contradiction):**

- Assume that property (u) holds.
- Then  $(\chi_n)$  is weakly Cauchy in  $J_p$  and has no weak limit as  $e_0 = (1, 1, \dots) \in J_p^{**} \setminus J_p$ .
- So there is sequence  $(y_n) \subset J_p$  with  $\sum_{n=1}^{\infty} y_n$  WUC such that  $\chi_n - \sum_{i=1}^n y_i \xrightarrow{w} 0$ .
- If  $\sum_{n=1}^{\infty} y_n$  converges unconditionally then it must do so to  $e_0 = (1, 1, \dots)$ , but this is not in  $J_p$ .
- By the Theorem above,  $J_p$  contains  $c_0$ . **Contradiction!**

# Pełczyński's Property (u)

This proof still depends on  $J_p$  not containing copies of  $c_0$ ; so a new idea is needed for a successful proof.

Instead of going for an abstract approach, we can view the proof as a simple game with concrete sequences:

- 1 We supply a weak Cauchy sequence  $(x_n)$ .
- 2 Our opponent counters with a sequence  $(y_n)$  such that  $(x_n - \sum_{i=1}^n y_i)_{n \in \mathbb{N}}$  is weakly null.
- 3 We win if we can find an  $f \in V_p^*$  such that  $\sum_{n=1}^{\infty} |\langle y_n, f \rangle|$  is divergent. If none exists, we lose.

If we can show that a winning strategy exists for us, this proves that  $V_p$  does not have Pełczyński's property (u).

# Pełczyński's Property (u) - Opening Serve

- As  $V_p$  has a shrinking basis for  $p > 1$ , a sequence is weak Cauchy if and only if it is bounded and  $(\langle x_n, f_k \rangle)_{n \in \mathbb{N}}$  converges for all  $k$ .
- If  $(x_n)$  is a weak Cauchy sequence that weakly converges then the conditions for Pełczyński's Property (u) to hold are trivially satisfied.
- So we need  $(x_n)$  not weakly convergent in  $V_p$ .
- A natural candidate for our sequence is, again, the bounded approximate identity  $x_n = \chi_n$ .

# Pełczyński's Property (u) - Return

- As  $V_p$  is a vector subspace of  $c_0$ , if  $(x_n)$  is weakly Cauchy in  $V_p$ , then it is in  $c_0$  too. Hence for all  $f \in I_1 = c_0^*$ , the sum  $\sum_{n=1}^{\infty} |\langle y_n, f \rangle|$  converges for all possible returned sequences  $(y_n)$ .
- To have any chance of winning, we must find  $f \in V_p^* \setminus I_1$ .
- A candidate functional, not defined on  $I_1$ :

$$\sum_n \frac{1}{n} f_n \notin V_p^*.$$

- Evaluation against  $\chi_n$  gives

$$\left\langle \chi_n, \sum_k \frac{1}{k} f_k \right\rangle = \sum_{k=1}^n \frac{1}{k} \rightarrow \infty$$

# Pełczyński's Property (u) - Return

- Choosing  $x_n = \chi_n$ , forces  $(y_n)$  to have weights in each coordinate eventually summing to 1. We want to choose  $f$  that picks out these large weights.
- Sum of alternating harmonic series converges

$$\sum_n \frac{(-1)^n}{n} = -\log 2,$$

but its absolute values, the harmonic series diverges

$$\sum_n \frac{1}{n} = \infty.$$

- We do have

$$\xi := \sum_n \frac{(-1)^n}{n} f_n \in V_p^*.$$

# Pełczyński's Property (u) - Match Point

Want to show that

$$\sum |\langle y_m, f \rangle| \quad (\star)$$

diverges for some choice of  $f$ .

- If  $y_n = e_n$  then we win with  $\xi$  as defined.
- If faced with a block basic sequence

$$y_n = \sum_{i=\sigma(n)}^{\sigma(n+1)-1} \alpha_i e_i,$$

(with increasing  $\sigma(n)$ ), then we win by playing:

$$\xi^\sigma := \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n} f_{\sigma(n)} \in V_p^*.$$

- We can ignore any terms of  $(y_n)$  and prove that for a subsequence,  $(\star)$  diverges.
- Add terms and show sum diverges. ( $|a| + |b| \geq |a + b|$ )

# Pełczyński's Property (u) - Match Point

We aim to exploit this fact by summing consecutive terms  $y_n$  to construct an 'approximate block basic sequence'  $(z_n)$ , with small weight on the initial co-ordinates and tail, and approximately one on the non-overlapping 'blocks'.

- Approximate blocks:  $(z_n)$
- Perfect blocks:  $(u_n)$
- These are (in some sense) close:

$$\|u_n - z_n\| < \epsilon.$$

Choosing these approximate blocks is a delicate process.

# Pełczyński's Property (u)

## Theorem

$V_p$  does not have Pełczyński's property (u).

## Corollary

$V_p$  doesn't embed in any space that has an unconditional basis.

## Conclusion

Hence  $V_p$  is not isomorphic to any  $S_q$  for  $q \geq 1$ .