

The James–Schreier Space

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Definitions

In this talk: $x = (\alpha_j) \subset \mathbb{R}^{\mathbb{N}}$;

$$\nu_p(x, A) := \left(\sum_{n \in A} |\alpha_n|^p \right)^{1/p}$$

Schreier norm: $\|x\|_{S_p} := \sup \{ \nu_p(x, A) : A \subseteq \mathbb{N} \text{ admissible} \}$

$$\mu_p(x, A) := \left(\sum_{j=1}^k |\alpha_{n_j} - \alpha_{n_{j+1}}|^p \right)^{1/p}$$

James norm: $\|x\|_{J_p} := \sup \{ \mu_p(x, A) : A \subseteq \mathbb{N} \}$

James–Schreier norm: $\|x\|_{V_p} := \sup \{ \mu_p(x, A) : A \subseteq \mathbb{N} \text{ permissible} \}$

Definitions

Two ways of defining J_p :

Definition 1

Let J_p be completion of c_{00} with respect to the J_p norm.

Definition 2

Let J_p be those sequences in c_0 which have finite J_p norm.

These two definitions describe the same space as

$$\|(I - P_n)x\|_{J_p} \rightarrow 0 \Leftrightarrow \|x\|_{J_p} < \infty,$$

that is, the unit vectors (e_i) are a Schauder basis of J_p in Definition 2.

Definitions

Definition - V_p

Let V_p be completion of c_{00} with respect to the $\|\cdot\|_{V_p}$ -norm.

Definition - W_p

Let W_p be those sequences in c_0 which have finite $\|\cdot\|_{V_p}$ -norm.

Here, the spaces V_p and W_p are not equal or isomorphic.

Equivalent Definition - V_p

Let V_p to be the closure of the the linear span of the basic sequence (e_i) in W_p with respect to the $\|\cdot\|_{V_p}$ -norm.

These two definitions of V_p are trivially equivalent. However, unlike the James space, now the basic sequence (e_i) , the basis by definition, of V_p , does not span W_p .

Definitions

Similarly for the Schreier space:

Definition - Z_p

Let Z_p be those sequences in c_0 which have finite $\|\cdot\|_{S_p}$ -norm.

Definition - S_p

Let S_p to be the closure of the the linear span of the basic sequence (e_i) in Z_p with respect to the S_p norm.

e.g. $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ has finite $\|\cdot\|_{S_1}$ -norm, so is in Z_1 , but is not in S_1 .

Claim: Z_p is the second dual of the Schreier space S_p .

Shrinking basis

Let (b_n) be a basis for a Banach space X .

Definition and Theorem

Let (f_n) be the sequence of **coordinate functionals** for (b_n) , that is, $f_n : \sum \alpha_m b_m \mapsto \alpha_n$; then each f_n is bounded.

Definition

A basis (b_n) is **shrinking** if and only if the coordinate functionals (f_n) are a basis for X^* .

Proposition

For $p > 1$ the unit vectors (e_i) are a shrinking basis for V_p .

Second Dual

- Define the natural map $\kappa : V_p \rightarrow V_p^{**}$ by $\kappa(x) : f_i \mapsto x_i$.
- The shrinking basis for V_p allows us to construct an isometric isomorphism from V_p^{**} to

$$X_{V_p} := \left\{ (\alpha_n) \subseteq \mathbb{C}^{\mathbb{N}} : \sup_m \left\| \sum_{n=1}^m \alpha_n e_n \right\|_{V_p} < +\infty \right\} \cong W_p \oplus \mathbb{C}1.$$

The James–Schreier Banach Algebra

For $1 \leq p < \infty$ the space X_{V_p} is a commutative $*$ -algebra under the norm $\|\cdot\|_{V_p}$ equipped with pointwise multiplication and pointwise complex conjugation as involution, and has separately continuous product.

The Banach space X_{V_p} is a commutative Banach $*$ -algebra with identity $e_0 = (1, 1, \dots)$ under the $\|\cdot\|_{V_p}$ -norm equipped with pointwise multiplication.

W_p is a $*$ -subalgebra of X_{V_p} .

V_p is a $*$ -ideal of W_p and X_{V_p} .

The James–Schreier Banach Algebra

We define χ_n to be $\sum_{i=1}^n e_i = (1, 1, \dots, 1, 0, \dots)$. The sequence (χ_n) is a bounded approximate identity of projections of $\|\cdot\|_{V_p}$ -norm 1 in the Banach $*$ -algebra V_p , and they are contained in c_{00} .

The commutative Banach $*$ -algebra V_p is weakly amenable, but not amenable.

V_p is Arens regular. Hence, the multiplier algebra of V_p , (V_p^{**}, \square) , X_{V_p} , $W_p \oplus \mathbb{C}1$ are all isometrically isomorphic.

James' Theorem

James' Theorem - 1950

A Banach space with an unconditional basis, is reflexive if and only if it has no embedded copies of c_0 or l_1 .

Corollary

The James space has no copies of c_0 or l_1 , but is not reflexive. So J_p has no unconditional basis.

But the James–Schreier space, V_p *does* contain copies of c_0 —it is c_0 -saturated! So a different approach is necessary.

Pełczyński's Property (u)

Pełczyński's Property (u) - 1958

A Banach space X has **Pełczyński's property (u)** if for all weak Cauchy sequences $(x_n) \subset X$ there exists a sequence $(y_n) \subset X$ such that for all $f \in X^*$

$$\left\langle x_n - \sum_{i=1}^n y_i, f \right\rangle \rightarrow 0 \text{ as } n \rightarrow \infty$$

and $\sum_{n=1}^{\infty} |\langle y_n, f \rangle|$ is convergent.

Pełczyński's Property (u)

Reminder: X has Pełczyński's property (u) if for all weak Cauchy sequences $(x_n) \subset X$ there exists a weakly unconditionally convergent (WUC) series $\sum_{n=1}^{\infty} y_n$ such that $(x_n - \sum_{i=1}^n y_i)_{n \in \mathbb{N}}$ is weakly null.

Every subspace of a space with Pełczyński's property (u) also has Pełczyński's property (u).

Every Banach space with an unconditional basis has property (u). In particular c_0 and S_p have it.

To show a Banach space does not have an unconditional basis it is enough to show it doesn't have property (u).

Pełczyński's Property (u) - James Space

Theorem - Bessaga and Pełczyński

Every weak unconditionally convergent (WUC) series in a Banach space X is unconditionally convergent if and only if X contains no copy of c_0 .

Proposition: J_p does not have Pełczyński's property (u).

Proof (by contradiction):

- Assume that property (u) holds.
- Then (χ_n) is weakly Cauchy in J_p and has no weak limit as $e_0 = (1, 1, \dots) \in J_p^{**} \setminus J_p$.
- So there is sequence $(y_n) \subset J_p$ with $\sum_{n=1}^{\infty} y_n$ WUC such that $\chi_n - \sum_{i=1}^n y_i \xrightarrow{w} 0$.
- If $\sum_{n=1}^{\infty} y_n$ converges unconditionally then it must do so to $e_0 = (1, 1, \dots)$, but this is not in J_p .
- By the Theorem above, J_p contains c_0 . **Contradiction!**

Pełczyński's Property (u)

This proof still depends on J_p not containing copies of c_0 ; so a new idea is needed for a successful proof.

Instead of going for an abstract approach, we can view the proof as a simple game with concrete sequences:

- 1 We supply a weak Cauchy sequence (x_n) .
- 2 Our opponent counters with a sequence (y_n) such that $(x_n - \sum_{i=1}^n y_i)_{n \in \mathbb{N}}$ is weakly null.
- 3 We win if we can find an $f \in V_p^*$ such that $\sum_{n=1}^{\infty} |\langle y_n, f \rangle|$ is divergent. If none exists, we lose.

If we can show that a winning strategy exists for us, this proves that V_p does not have Pełczyński's property (u).

Pełczyński's Property (u) - Opening Serve

- As V_p has a shrinking basis for $p > 1$, a sequence is weak Cauchy if and only if it is bounded and $(\langle x_n, f_k \rangle)_{n \in \mathbb{N}}$ converges for all k .
- If (x_n) is a weak Cauchy sequence that weakly converges then the conditions for Pełczyński's Property (u) to hold are trivially satisfied.
- So we need (x_n) not weakly convergent in V_p .
- A natural candidate for our sequence is, again, the bounded approximate identity $x_n = \chi_n$.

Pełczyński's Property (u) - Return

- As V_p is a vector subspace of c_0 , if (x_n) is weakly Cauchy in V_p , then it is in c_0 too. Hence for all $f \in I_1 = c_0^*$, the sum $\sum_{n=1}^{\infty} |\langle y_n, f \rangle|$ converges for all possible returned sequences (y_n) .
- To have any chance of winning, we must find $f \in V_p^* \setminus I_1$.
- A candidate functional, not defined on I_1 :

$$\sum_n \frac{1}{n} f_n \notin V_p^*.$$

- Evaluation against χ_n gives

$$\left\langle \chi_n, \sum_k \frac{1}{k} f_k \right\rangle = \sum_{k=1}^n \frac{1}{k} \rightarrow \infty$$

Pełczyński's Property (u) - Return

- Choosing $x_n = \chi_n$, forces (y_n) to have weights in each coordinate eventually summing to 1. We want to choose f that picks out these large weights.
- Sum of alternating harmonic series converges

$$\sum_n \frac{(-1)^n}{n} = -\log 2,$$

but its absolute values, the harmonic series diverges

$$\sum_n \frac{1}{n} = \infty.$$

- We do have

$$\xi := \sum_n \frac{(-1)^n}{n} f_n \in V_p^*.$$

Pełczyński's Property (u) - Match Point

Want to show that

$$\sum |\langle y_m, f \rangle| \quad (\star)$$

diverges for some choice of f .

- If $y_n = e_n$ then we win with ξ as defined.
- If faced with a block basic sequence

$$y_n = \sum_{i=\sigma(n)}^{\sigma(n+1)-1} \alpha_i e_i,$$

(with increasing $\sigma(n)$), then we win by playing:

$$\xi^\sigma := \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n} f_{\sigma(n)} \in V_p^*.$$

- We can ignore any terms of (y_n) and prove that for a subsequence, (\star) diverges.
- Add terms and show sum diverges. ($|a| + |b| \geq |a + b|$)

Pełczyński's Property (u) - Match Point

We aim to exploit this fact by summing consecutive terms y_n to construct an 'approximate block basic sequence' (z_n) , with small weight on the initial co-ordinates and tail, and approximately one on the non-overlapping 'blocks'.

- Approximate blocks: (z_n)
- Perfect blocks: (u_n)
- These are (in some sense) close:

$$\|u_n - z_n\| < \epsilon.$$

Choosing these approximate blocks is a delicate process.

Pełczyński's Property (u)

Theorem

V_p does not have Pełczyński's property (u).

Corollary

V_p doesn't embed in any space that has an unconditional basis.

Conclusion

Hence V_p is not isomorphic to any S_q for $q \geq 1$.