Approximate projectivity for Banach modules

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DEFINITIONS

A Banach module X over a Banach algebra A is called *projective* if $\forall$ admissible lifting problem (i.e., an admissible epimorphism $\varepsilon : Y' \to Y$ and a morphism $\psi : X \to Y$) $\exists$ an A-module morphism $\rho : X \to Y'$ s.t. the diagram

$$\begin{array}{ccc} & & Y' \\ & \overset{\rho}{\nearrow} & \big\downarrow \varepsilon \\ X & \xrightarrow[\psi]{} & Y \end{array} \qquad (1)$$

commutes.

X if called *flat* if $? \widehat{\otimes}_A X$ sends $\forall$ admissible sequence of Banach modules to exact sequence.

Analytic approach:

Flatness + AP $\Rightarrow$ approximate projectivity, i.e.

Theorem 1 (Aristov, 2005) *X is flat and has the approximation property $\Rightarrow \forall$ admissible lifting problem $\exists$ a net of bounded linear maps $\rho_\lambda : X \to Y'$ s.t.*
(1) *Diagram (1) commutes,*
(2) $[a \cdot \rho_\lambda(x) - \rho_\lambda(a \cdot x)] \to 0$ *uniformly by a and x on compact subsets of A and X.*

We say that X is *approximately projective* if it satisfies to the conclusion of Theorem 1.

In the initial definition:

(1') Diagram (1) approximately commutes w.r.t. compact sets.

The AP is not necessary: every module over an amenable algebra is approximately projective.

Why we use compact subsets? They are important in theory of projective tensor products of Banach spaces (and modules).

Theorem 2 (Grothendieck, Reformulation) *Let E and F be Banach spaces, and K be a compact subset in $E \widehat{\otimes} F$. Then $\exists$ compact subsets K' in E and K'' in F s.t. $\forall$ element of K has the form $\sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n$ where $e_n \in K'$, $f_n \in K''$, $\lambda_n \in \mathbb{C}$ and $\sum |\lambda_n| \leq 1$.*

In fact, an element in the tensor product corresponds to compact sets in E and F.

More general: Let $\mathcal{F} = (\mathcal{F}_A, \mathcal{F}_X)$ be classes of bounded subsets of A and X resp. (another good choice is class of finite sets).

Definition 3 *We say that X is $\mathcal{F}$-*approximately projective *if $\forall$ admissible lifting problem $\exists$ a net of bounded linear maps $\rho_\lambda : X \to Y'$ s.t.*
(1) *Diagram (1) commutes,*
(2) $[a \cdot \rho_\lambda(x) - \rho_\lambda(a \cdot x)] \to 0$ *uniformly by a and x on subsets from $\mathcal{F}_A$ and $\mathcal{F}_X$.*

Fix $\mathcal{F}_A$ and $\mathcal{F}_X$ $\forall$ B.m. X.

Definition 4 *We say that* X *is* strongly $\mathcal{F}$-approximately projective *if* $\forall$ *every admissible sequence*

$$0 \longleftarrow X \longleftarrow P \overset{\iota}{\longleftarrow} Z \longleftarrow 0, \qquad (2)$$

of Banach A*-modules, where* P *is projective,* $\exists$ *a net* $\varphi_\lambda \colon P \to Z$ *s.t.*

$$\varphi_\lambda \iota(z) \to z$$

uniformly on subsets from $\mathcal{F}_Z$*.*

In fact, the existence of only one such a sequence implies strong $\mathcal{F}$-approximate projectivity.

Theorem 5 (Natural) *If* $\mathcal{F}$ *is invariant w.r.t. bounded linear operators and compatible with the multiplication then every strongly* $\mathcal{F}$*-approximately projective module is* $\mathcal{F}$*-approximately projective.*

MOTIVATION

Pure Algebra

Let $0 \leftarrow X \leftarrow P \stackrel{\iota}{\leftarrow} Z \longleftarrow 0$ *be an exact sequence of modules over an unital ring such that* P *is projective. Then* X *is flat iff for every* $z_1, \ldots, z_n \in Z$ *there is a morphism of modules* $\varphi : P \to Z$ *such that* $\varphi\iota(z_k) = z_k$ *for all* k (Villamayor $\approx$ 1960)

In fact, this is a point-wise convergence.

Analisys

Theorem 6 (Main) *If* $\mathcal{C}$ *is a class of compact subsets then every* $\mathcal{C}$*-approximately projective module is strongly* $\mathcal{C}$*-approximately projective.*

May be approximate projectivity is better analogue of flatness in the B.a. context.

How approximate projectivity compatible with standard homological constructions (derived functors, cohomology groups e.t.c)?

DERIVED FUNCTORS

Construction:

$\mathrm{Ext}^n_A(X,Y)$ ($n \in \mathbf{N}$) is the nth derived functor from ${}_A\mathrm{h}(\,\cdot\,, Y)$, i.e. the nth homology of the complex ${}_A\mathrm{h}(P_\bullet(X), Y)$, where $0 \leftarrow X \leftarrow P_\bullet(X)$ is a projective resolution of X. When $P_\bullet(X)$ is the standard resolution (in par. $P_n(X) = A_+ \widehat{\otimes} A^{\widehat{\otimes} n} \widehat{\otimes} X$). Then we have a c.d.

$$\begin{array}{ccccccccc} 0 \to & {}_A\mathrm{h}(P_0(X),Y) & \to & {}_A\mathrm{h}(P_1(X),Y) & \to & {}_A\mathrm{h}(P_2(X),Y) & \to \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathcal{B}(X,Y) & \longrightarrow & \mathcal{B}(A\times X,Y) & \longrightarrow & \mathcal{B}(A^2\times X,Y) & \to \end{array}$$

The low line is the *standard complex* with the differential

$$d_n(\psi)\colon (a_1,\dots,a_{n+1},x) \mapsto a_1\cdot\psi(a_2,\dots,a_{n+1},x) + \sum_{k=1}^{n} (-1)^k \psi(a_1,\dots,a_k a_{k+1},\dots,a_{n+1},x) + (-1)^{n+1}\psi(a_1,\dots,a_n,a_{n+1}\cdot x).$$

Now we endow every member of the standard complex with the topology of uniformly polylinear convergence w.r.t. $\mathcal{F}$.

Definition 7 *Denote by* $\mathrm{Ext}^n_A(X,Y)_{\mathcal{F}}$ *the* n*th homology of* $\mathcal{B}(A^\bullet \times X, Y)_{\mathcal{F}}$.

Theorem 8 *Let* X *be a left Banach* A*-module. Then* X *is* $\mathcal{F}$*-approximately projective iff* $\forall$ *left Banach* A*-module* Y *the topology in* $\mathrm{Ext}^1_A(X,Y)_{\mathcal{F}}$ *is trivial iff the topology* $\mathrm{Ext}^n_A(X,Y)_{\mathcal{F}}$ *is trivial* $\forall n$.

The idea to consider the triviality of topology in $\mathrm{Ext}^n_A(X,Y)_{\mathcal{F}}$ to define some class of modules belongs to Pirkovskii (2005, unpublished notes, the case of finite sets).

Recall that X is projective iff $\forall$ left Banach A-module Y $\mathrm{Ext}^1_A(X,Y) = 0$ iff $\mathrm{Ext}^n_A(X,Y) = 0$ $\forall n$ (Helemskii).

There is a similar characterization of flat modules.

(Moreover, X is projective iff X is approximately projective w.r.t. bounded sets iff $\forall$ left Banach A-module Y the topology in $\mathrm{Ext}^n_A(X,Y)$ is trivial $\forall n$ (Pirkovskii, 2006))

But $\mathrm{Ext}^n_A(X,Y)_{\mathcal{F}}$ is not a derived functor in the strict sense.

Definition 9 *Denote by* $\mathsf{R}'\mathsf{h}^n(X,Y)_{\mathcal{F}}$ *the* n*th derived functor from* ${}_A\mathsf{h}(\,\cdot\,,Y)_{\mathcal{F}}$*, i.e. homology of* ${}_A\mathsf{h}(P_\bullet,Y)_{\mathcal{F}}$*.*

It is possible to consider d.f. from ${}_A\mathsf{h}(X,\,\cdot\,)_{\mathcal{F}}$ and ${}_A\mathsf{h}(\,\cdot\,,\,\cdot\,)_{\mathcal{F}}$ but there is no reason for them to coincide as in the norm-topology case!

Theorem 10 *Let* X *be a left Banach* A*-module. Then* X *is strongly* $\mathcal{F}$*-approximately projective iff* $\forall$ *left Banach* A*-module* Y *the topology in* $\mathsf{R}'\mathsf{h}^1(X,Y)_{\mathcal{F}}$ *is trivial iff* $\forall$ *left Banach* A*-module* Y *the topology in* $\mathsf{R}'\mathsf{h}^n(X,Y)_{\mathcal{F}}$ *is trivial* $\forall n$*.*

Gronbaek (1995) and Y.Zhang (1999) considered another approach to approximate projectivity based on morphisms and approximate commutativity of Diagram (1). But I do not find a direct connection with general homological constructions in this case.

EXAMPLES

Proposition 11 *Let J be a left closed ideal in a Banach algebra A s.t. $A_+ \to A_+/J$ is admissible (i.e. J is complemented as a Banach space). Then A_+/J is approximately projective iff J admits a stable right a.i. (i.e. $au_\lambda \to a$ uniformly on compact subsets).*

Denote by $\mathsf{R}''\mathsf{h}^n(X,Y)_{\mathcal{F}}$ the nth derived functor from ${}_A\mathsf{h}(X,\cdot\,)_{\mathcal{F}}$ (by injective resolutions!)

There is a continuous bijective linear operator

$$\mathsf{R}''\mathsf{h}^n(X,Y)_{\mathcal{F}} \longrightarrow \mathsf{R}'\mathsf{h}^n(X,Y)_{\mathcal{F}}.$$

In the compact subsets case:

$$\mathsf{R}'\mathsf{h}^n(X,Y)_{\mathcal{C}} \cong \mathsf{Ext}^n_A(X,Y)_{\mathcal{C}}$$

naturally (comp. Main Theorem)

$A = c_0$, $\mathbb{C} = A_+/A$.

Proposition 12 *The topology in* $\mathsf{R}'\mathsf{h}^1(\mathbb{C}, c_0)_{\mathcal{C}}$ *is trivial.*

Proposition 13 $\mathsf{R}''\mathsf{h}^1(\mathbb{C}, c_0)_{\mathcal{C}} \cong c_b/c_0$.

Corollary 14 $\mathsf{R}''\mathsf{h}^1(\mathbb{C}, c_0)_{\mathcal{C}} \to \mathsf{R}'\mathsf{h}^1(\mathbb{C}, c_0)_{\mathcal{C}}$ *is not a topological isomorphism.*

APPROXIMATE CONTRACTIBILITY

Ghahramani & Loy (JFA 2004) call a Banach algebra A *approximately contractible* if $\forall$ continuous derivation $D\colon A \to Y$ $\exists$ a net $(y_\lambda) \subset Y$ s.t.

$$(a \cdot y_\lambda - y_\lambda \cdot a) \to D(a)$$

uniformly on finite sets. (This is equivalent to more general approximate amenability (Ghahramani, Loy & Zhang).)

We say that A is *$\mathcal{F}$-approximately contractible* if

$$(a \cdot y_\lambda - y_\lambda \cdot a) \to D(a)$$

uniformly on subsets from $\mathcal{F}_A$.

Theorem 15 *A Banach algebra is $\mathcal{F}$-approximately contractible iff it is an $\mathcal{F}$-approximately projective bimodule over itself.*

Theorem 16 *Every Banach module over a $\mathcal{C}$-approximately contractible Banach algebra is approximately projective.*

(I am sure that this is not true for G&L definition but I do not know counterexamples.)

Corollary 17 *Every $\mathcal{C}$-approximately contractible Banach algebra admits both left and right stable a.i.*

Sf.: A contractible B.a. admits identity;
an amenable B.a. admits b.a.i.

Theorem 18 *Every amenable Banach algebra is $\mathcal{C}$-approximately contractible.*