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Extremely flat c_0-modules and the tensor product of $L_p(\Omega, \mu)$ and $L_p(\Omega, \nu)$ over $C_0(\Omega)$

ABSTRACT. Let A be a normed algebra, $\mathcal{K}$ some class of right normed A-modules. A normed A-module Z is called *extremely flat relative to* $\mathcal{K}$ if, for every isometric morphism $i : X \to Y; X, Y \in \mathcal{K}$, the projective module tensor product $i \otimes_A 1 : X \otimes_A Z \to Y \otimes_A Z$ is also isometric. The investigation of modules over some highly non-commutative C^*-algebras, extremely flat relative to the class of semi-Ruan modules, was initiated by Helemskii and then considerably developed by Wittstock.

These results aroused the interest to extremely flat modules over the opposite, in a sense, class of algebras, that is commutative C^*-algebras. We began with the simplest of these, the algebra c_0 of vanishing sequences. The following theorem describes extremely flat modules within a certain reasonable class of c_0-modules.

For a given normed c_0-module Z we set $Z_n := \{p^n \cdot z; z \in Z\}$, where p^n is the n-th "ort" in c_0. We call Z *homogeneous*, if, for $z', z'' \in Z$, we have $\|z'\| = \|z''\|$ provided $\|p^n \cdot z'^n \cdot z''\|$ for all n.

Theorem. *Let $\mathcal{K}$ be a class of contractive essential homogeneous c_0-modules. Then a module $Z \in \mathcal{K}$ is extremely flat relative to $\mathcal{K}$ if, and only if, for every $n = 1, 2, .., Z_n$ is a dense normed subspace of $L_1(\Omega)$ for some measure space Ω.*

For example, every contractive essential *sequence* module over c_0 consisting of sequences and endowed with coordinate-wise outer multiplication, is extremely flat relative to $\mathcal{K}$.

What about general commutative C^*-algebras, that is $C_0(\Omega)$, with Ω an arbitrary locally compact space? Presently, we can only compute tensor products of some particular $C_0(\Omega)$-modules, namely those of L_p-type.

Let μ and ν be regular Borel measures on Ω. Take the canonical decomposition $\nu = f\mu + \mu_0$, where $f \in L_1^{loc}(\Omega)$, $\mu_0 \perp \mu$, and consider the measure $\lambda := f^{\frac{p}{p+q}}\mu$. Set $\mathcal{A} := \{t \in \Omega : \lambda\{t\} > 0\}$ and denote by $l_1(\mathcal{A}, r, \lambda)$ the space of functions z on $\mathcal{A}$ such that $\|z\| := \sum\{|z(t)|(\lambda\{t\})^{\frac{1}{r}}; t \in \mathcal{A}\} < \infty$. Finally, let $\otimes_\Omega$ denote the completed projective module tensor product over $C_0(\Omega)$.

Theorem. *Let $p, q \in [1, \infty)$, and r be determined by $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then*

$$L_p(\Omega, \mu) \otimes_\Omega L_q(\Omega, \nu) = \begin{cases} L_r(\Omega, \lambda), & if\ r \geq 1 \\ l_1(\mathcal{A}, r, \lambda) \text{ (e.g., } \mathbf{0}, \text{if } \mu \text{ or } \nu \text{ is non} - \text{atomic)}, & if\ r < 1. \end{cases}$$