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CLIFFORD OPERATOR CALCULUS: HOMOLOGY, COHOMOLOGY, AND APPELL SYSTEMS

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Abstract. *Appell systems can be interpreted as polynomial solutions of generalized heat equations, and in probability theory they have been used to obtain non-central limit theorems. The natural grade-decomposition of a Clifford algebra of arbitrary signature lends it a natural Appell system decomposition. Motivated by evolution equations on Clifford algebras, combinatorial raising and lowering operators defined on Clifford algebras lead to a theory of invertible left- and right- Clifford Appell systems. These operators further allow one to define chains and cochains of vector spaces corresponding to the algebra's Appell system decomposition. The associated homology and cohomology groups can then be computed, and long exact sequences of underlying vector spaces can be derived. Operator calculus methods also readily lend themselves to convenient symbolic computations; for example, kernels of lowering and raising operators can be explicitly computed with Mathematica.*

1 INTRODUCTION

Appell systems can be interpreted as polynomial solutions of generalized heat equations. In probability theory, they are also used to obtain non-central limit theorems. Their analogues have been defined on Lie groups [3], the Schrödinger algebra [2], and quantum groups [1]. Clifford Appell systems are natural objects of interest for constructing solutions of Clifford evolution equations.

The current authors first defined general Appell systems within a Clifford algebra of arbitrary signature in [6]. The operator calculus (OC) appearing in that work was subsequently used in a treatment of operator homology and cohomology in Clifford algebras [10]. More recently, OC methods were extended to commutative subalgebras of the Grassmann exterior algebra referred to herein as "zeon" algebras and used to give an OC formulation of partition-dependent stochastic measures [11].

The authors' OC approach is implicit in a number of earlier works in graph theory (cf. [15], [12]). Operator calculus on zeon algebras provides the common context for relating graph theory and quantum random variables [7]. Moreover, the OC approach has been advantageous in developing a graph-theoretic construction of stochastic integrals in Clifford algebras of arbitrary signature [14].

Raising and lowering operators can be used to define chains and cochains on vector spaces underlying a Clifford algebra of arbitrary signature. The associated homology and cohomology groups can then be computed, and long exact sequences of vector spaces can be derived.

2 CLIFFORD ALGEBRA GENERALITIES

Let $n \in \mathbb{N}$, and let V be an n-dimensional, real inner product space over equipped with a nondegenerate quadratic form Q. The *Clifford algebra* $\mathcal{C}\ell_Q(V)$ is the real associative algebra generated by the vectors $\mathbf{x} \in V$, along with the unit scalar 1 subject to:

$$\mathbf{x}^2 = Q(\mathbf{x}). \tag{1}$$

Recalling the real exterior algebra $\bigwedge V$, associate with Q the symmetric bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle_Q = \frac{1}{2}\left[Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})\right], \tag{2}$$

and extend to simple k-vectors in $\bigwedge^k V$ by

$$\langle \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_k, \mathbf{y}_1 \wedge \mathbf{y}_2 \wedge \cdots \wedge \mathbf{y}_k \rangle_Q = \det \langle \mathbf{x}_i, \mathbf{y}_j \rangle_Q . \tag{3}$$

This inner product extends linearly to all of $\bigwedge^k V$ and by orthogonality to $\bigwedge V$.

The Q-inner product and exterior product extend to $\mathcal{C}\ell_Q(V)$ via the canonical vector space isomorphism. The *left contraction* operator is defined by (cf. [5, Chapter 14])

$$\mathbf{x} \lrcorner \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle_Q \;\; \forall \mathbf{x}, \mathbf{y} \in V; \tag{4}$$

$$\mathbf{x} \lrcorner (u \wedge v) = (\mathbf{x} \lrcorner u) \wedge v + \hat{u} \wedge (\mathbf{x} \lrcorner v), \;\; \forall u, v \in \bigwedge V, \mathbf{x} \in V; \tag{5}$$

$$(u \wedge v) \lrcorner w = u \lrcorner (v \lrcorner w), \;\; \forall u, v, w \in \bigwedge V. \tag{6}$$

Here, $\hat{u}$ denotes the *grade involution* of u.

Given a basis $\mathcal{B} = \{\mathbf{e}_i : 1 \le i \le n\}$ of V satisfying $Q(\mathbf{e}_i) = \pm 1$ for each i and $\langle \mathbf{e}_i, \mathbf{e}_j \rangle_Q = 0$ for $i \neq j$, the algebra is generated by $\mathcal{B}$ together with the unit scalar. Denote the n-set $\{1, \ldots, n\}$ by $[n]$, and denote the associated *power set* by $2^{[n]}$. Adopting *multi-index notation*, the ordered product of generators is denoted

$$\prod_{i \in I} \mathbf{e}_i = \mathbf{e}_I, \tag{7}$$

for any subset $I \subseteq [n]$, also denoted $I \in 2^{[n]}$.

These products of generators are referred to as basis *blades* for the algebra. The *grade* of a basis blade is defined to be the cardinality of its multi-index. An arbitrary element $u \in \mathcal{C}\ell_Q(V)$ has a canonical basis blade decomposition of the form

$$u = \sum_{I \subseteq [n]} u_I \, \mathbf{e}_I, \tag{8}$$

where $u_I \in \mathbb{R}$ for each multi-index I. The *grade-k part* of $u \in \mathcal{C}\ell_Q(V)$ is then naturally defined by

$$\langle u \rangle_k := \sum_{|I|=k} u_I \mathbf{e}_I. \tag{9}$$

An arbitrary element $u \in \mathcal{C}\ell_Q(V)$ is said to be *homogeneous of grade k* if

$$\langle u \rangle_k \neq 0, \;\; \text{and}$$
$$\langle u \rangle_\ell = 0, \;\; \forall \ell \neq k.$$

3 CLIFFORD APPELL SYSTEMS

Following the formalism of Feinsilver, Kocik, and Schott [2], the space of polynomials with degree not exceeding n can be considered as the space of solutions, $\mathscr{Z}_n$, to the equation $D^{n+1}\psi = 0$, where D is the differentiation operator. In this context, an *Appell system* is a sequence of nonzero polynomials satisfying two conditions:

1. $\psi_n \in \mathscr{Z}_n$, $\forall n \ge 0$, and
2. $D\psi_n = \psi_{n-1}$, $\forall n \ge 1$.

A simple example of an Appell system is to define $\psi_n = x^n/n!$ with $D = d/dx$. Other examples of Appell systems include shifted moment sequences

$$\psi_n(x) = \int_{-\infty}^{\infty} (x+y)^n p(dy) \tag{10}$$

where p is a probability measure on $\mathbb{R}$ with all moments being finite. This includes the Hermite polynomials,

$$H_n(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} (x+y)^n e^{-y^2/2t} dy, \;\; t > 0, \tag{11}$$

for the Gaussian case. These polynomials are the solutions of the heat equation

$$\partial_t f(x,t) = \frac{1}{2} \partial_x^2 f(x,t)$$

with $\lim_{t\to 0} f(x,t) = x^n$.

Clifford Appell systems are constructed using combinatorial raising and lowering operators defined on $\mathcal{C}\ell_Q(V)$ in terms of exterior products and contractions. These operators provide the mechanisms by which Clifford monomials (blades) are "raised" from grade k to grade $k+1$ or "lowered" from grade k to grade $k-1$. Such raising and lowering operators can also be regarded as fermion creation and annihilation operators in the sense of quantum mechanics.

Definition 3.1. A *homogeneous (left) Clifford Appell system* is a pair $\Psi = (\{\psi_k\}, \{\Lambda_{\mathbf{x}_k}\})$, where $\{\psi_k\} \subset \mathcal{C}\ell_Q(V)$, and $\{\Lambda_{\mathbf{x}_k}\}$ is a collection of generalized (left) lowering operators associated with a sequence of non-null Clifford vectors $(\mathbf{x}_k)$ such that

1. for each k, either $\psi_k = 0$ or ψ_k is homogeneous of grade k, and

2. $\Lambda_{\mathbf{x}_k}\psi_k = \psi_{k-1}$ for each k.

The operators $\Lambda_{\mathbf{x}}$ appearing here are defined in terms of left contraction. In particular, for $\mathbf{x} \in V$ and $u \in \mathcal{C}\ell_Q(V)$,

$$\Lambda_{\mathbf{x}}\, u := \frac{\partial}{\partial \mathbf{x}} u := \mathbf{x} \lrcorner u. \tag{12}$$

One analogously defines right lowering operators using right contraction.

One similarly defines a *homogeneous right Clifford Appell system* in terms of generalized right lowering operators.

Definition 3.2. The *order* of Ψ is defined as the maximum grade among the elements of $\{\psi_k\}$. The system Ψ is said to be of *rank* r if $\{\psi_k\}$ contains $r+1$ nonzero elements .

This definition of rank follows naturally from Appell systems of polynomials of a single variable since a polynomial's degree is always one greater than its number of nonzero derivatives.

Remark 3.3. In $\mathcal{C}\ell_{n,0}$, which is canonically isomorphic to the n-particle fermionic Fock space, the lowering operator maps monomials representing k-particle systems to monomials representing $(k-1)$-particle systems. In other words, the lowering operator acts as an annihilation operator.

To ensure $\psi_k \neq 0$ for all $1 \leq k \leq n$, appropriate conditions on the sequence $(\mathbf{x}_k)$ are addressed in the next proposition. Note that in any homogeneous Clifford Appell system associated with an n-dimensional vector space V, ψ_n is associated with a scalar multiple of the pseudoscalar $\psi_n = \alpha\, \mathbf{e}_{[n]}$.

Proposition 3.4. *Let $\Psi = (\{\psi_k\}, \{\Lambda_{\mathbf{x}_k}\})$ be an order-n homogeneous Clifford Appell system. The rank of Ψ is m if and only if the set $X = \{\mathbf{x}_{n-m+1}, \ldots, \mathbf{x}_n\}$ is linearly-independent; i.e.,* $\operatorname{rank}(X) = m$.

Proof. Omitted. □

Note that Proposition 3.4 guarantees the existence of an order-n, rank-n homogeneous Clifford Appell system associated with any basis of the vector space V generating $\mathcal{C}\ell_Q(V)$. The construction algorithm is straightforward:

1. Order the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V.

2. Set $\psi_n = \alpha \mathbf{e}_{[n]}$ for some scalar α.

3. Set $\psi_{k-1} = \mathbf{v}_k \lrcorner \psi_k$ for each $k = n, n-1, \ldots, 1$.

On the other hand, Proposition 3.4 does not guarantee the existence of such a system in infinite-dimensional Clifford algebras. Moreover, it does not address the construction of *invertible Appell systems* as defined below.

The construction of Clifford Appell systems is aided by defining the following special subclass of systems.

Definition 3.5. An *invertible homogeneous (left) Clifford Appell system* is a triple $\Psi = (\{\psi_k\}, \{\Lambda_{\mathbf{x}_k}\}, \{\Xi_{\mathbf{x}_k}\})$, where $\{\psi_k\} \subset \mathcal{C}\ell_Q(V)$, $\{\Lambda_{\mathbf{x}_k}\}$ is a collection of (left) lowering operators, and $\{\Xi_{\mathbf{x}_k}\}$ is a collection of (left) raising operators such that

1. for each k, ψ_k is homogeneous of grade k, and
2. $\Lambda_{\mathbf{x}_k}\psi_k = \psi_{k-1}$ for each k, and
3. $\Xi_{\mathbf{x}_k}\psi_{k-1} = \psi_k$ for each k.

The operators $\Xi_{\mathbf{x}}$ appearing here are analogous to Clifford integrals and are defined in terms of the exterior product. In particular, for $\mathbf{x} \in V$ and $u \in \mathcal{C}\ell_Q(V)$,

$$\Xi_{\mathbf{x}}\, u := \int u\, d\mathbf{x} := \mathbf{x} \wedge u. \tag{13}$$

Right lowering operators are defined using right exterior multiplication by $\mathbf{x}$.

Theorem 3.6 (Solutions)**.** *Let $\mathbf{x}$ be a non-null Clifford vector, and let ψ_{k-1} be a grade-$(k-1)$ Clifford multivector. Then, the Clifford equation*

$$\frac{\partial}{\partial \mathbf{x}}\psi_k = \psi_{k-1} \tag{14}$$

has a solution ψ_k if and only if $\dfrac{\partial}{\partial \mathbf{x}}\psi_{k-1} = 0$. Moreover, a solution of (14) *is the homogeneous grade-k multivector given by $\psi_k = \dfrac{1}{\mathbf{x}^2}\displaystyle\int \psi_{k-1}\, d\mathbf{x}$.*

Proof. A detailed proof can be found in [13]. □

Corollary 3.7 (Appell System Solutions)**.** *Let $\{\mathbf{x}_k : 1 \leq k \leq m\}$ be an ordered collection of vectors, orthogonal with respect to the quadratic form Q. Let ψ_0 be a scalar. Then, setting*

$$\psi_k := \frac{1}{{\mathbf{x}_k}^2}\int \psi_{k-1}\, d\mathbf{x}_k \tag{15}$$

for each $k = 1, \ldots, m$ gives an order-m, rank-m invertible homogeneous Clifford Appell system $\Psi = (\{\psi_k\}, \{\Lambda_{\mathbf{x}_k}\}, \{\Xi_{\mathbf{x}_k}\})$.

Example 3.8. The Clifford algebra $\mathcal{C}\ell_{3,3}$ is generated by orthogonal vectors $\{\mathbf{e}_i : 1 \leq i \leq 6\}$ satisfying ${\mathbf{e}_i}^2 = 1$ for $i = 1, 2, 3$ and ${\mathbf{e}_i}^2 = -1$ for $i = 4, 5, 6$. The following collection of orthogonal non-null vectors was constructed in $\mathcal{C}\ell_{3,3}$ using Mathematica:

```
Vector sequence (x_k):
x_1 = 0.995564 e_{1} + 0.0932456 e_{2} -
  0.00546402 e_{3} - 0.00554477 e_{4} - 0.00976338 e_{5} - 0.00103628 e_{6}
x_2 = -0.0879225 e_{1} + 0.960669 e_{2} - 0.114892 e_{3} - 0.11659 e_{4} - 0.205294 e_{5} - 0.0217898 e_{6}
x_3 =
 -0.00580452 e_{1} + 0.12202 e_{2} + 0.992415 e_{3} - 0.00769708 e_{4} - 0.0135532 e_{5} - 0.00143853 e_{6}
x_4 =
 0.00625123 e_{1} - 0.131411 e_{2} + 0.00770043 e_{3} + 1.02299 e_{4} + 0.0404897 e_{5} + 0.00429756 e_{6}
x_5 =
 0.0106783 e_{1} - 0.224475 e_{2} + 0.0131538 e_{3} - 0.0133482 e_{4} + 1.06916 e_{5} + 0.00734107 e_{6}
x_6 = 0.00103727 e_{1} - 0.0218051 e_{2} +
  0.00127774 e_{3} - 0.00129662 e_{4} - 0.00228312 e_{5} + 1.00071 e_{6}
```

By defining $\psi_0 := 1$ and proceeding inductively by setting $\psi_k := \Xi_{\mathbf{x}_k}\psi_{k-1}$ for $k = 1, \ldots, n$, the following invertible homogeneous Clifford Appell system was obtained:

```
Appell system:
ψ_0 = 1
ψ_1 = 0.995818 e_{1} + 0.0932693 e_{2} -
  0.00546541 e_{3} - 0.00554618 e_{4} - 0.00976586 e_{5} - 0.00103654 e_{6}
ψ_2 = -1.08703 e_{1,2} + 0.129441 e_{1,3} + 0.131354 e_{1,4} + 0.231291 e_{1,5} +
  0.0245491 e_{1,6} + 0.0061575 e_{2,3} + 0.00624851 e_{2,4} + 0.0110025 e_{2,5} + 0.0011678 e_{2,6}
ψ_3 = -1.09509 e_{1,2,3} - 0.00770043 e_{1,2,4} -
  0.0135591 e_{1,2,5} - 0.00143916 e_{1,2,6} - 0.131411 e_{1,3,4} - 0.231391 e_{1,3,5} -
  0.0245598 e_{1,3,6} - 0.00625123 e_{2,3,4} - 0.0110073 e_{2,3,5} - 0.00116831 e_{2,3,6}
ψ_4 = -1.06994 e_{1,2,3,4} - 0.0133482 e_{1,2,3,5} -
  0.00141678 e_{1,2,3,6} + 0.0131538 e_{1,2,4,5} + 0.00139614 e_{1,2,4,6} +
  0.224475 e_{1,3,4,5} + 0.0238257 e_{1,3,4,6} + 0.0106783 e_{2,3,4,5} + 0.00113339 e_{2,3,4,6}
ψ_5 = 1.00071 e_{1,2,3,4,5} + 0.00228312 e_{1,2,3,4,6} - 0.00129662 e_{1,2,3,5,6} +
  0.00127774 e_{1,2,4,5,6} + 0.0218051 e_{1,3,4,5,6} + 0.00103727 e_{2,3,4,5,6}
ψ_6 = 1. e_{1,2,3,4,5,6}
```

Many properties of lowering and raising operators on $\mathcal{C}\ell_Q(V)$ can be derived. The interested reader is directed to [10] and the preprint [13] for more details.

3.1 Remarks on Clifford Evolution Equations

Returning to solutions of evolution equations, non-constant discrete processes on Clifford algebras are described by evolution equations of the form

$$\partial_t u(k) = \Lambda_{\mathbf{x}_k} u(k), \tag{16}$$
$$\partial_t u(k) = \Xi_{\mathbf{x}_k} u(k), \text{ and} \tag{17}$$
$$\partial_t u(k) = (\Lambda_{\mathbf{x}_k} + \Xi_{\mathbf{x}_k}) u(k). \tag{18}$$

That is, the differential Δu at the k^{th} time step

$$\partial_t u(k) := u(k) - u(k-1) \tag{19}$$

in the discrete process is given by the action of generalized raising and lowering operators associated with a vector-valued process $(\mathbf{x}_k)$ in the vector space V generating $\mathcal{C}\ell_Q(V)$. The Clifford-valued process then becomes an additive process of the general form

$$u(k) := \sum_{\ell=0}^{k} \psi_{j_\ell}, \tag{20}$$

where (j_ℓ) is a sequence of Appell indices determined by the choice of raising, lowering, or raising+lowering.

The current authors have considered both time-homogeneous discrete processes (cf. [8]) and dynamic processes (cf. [9]) associated with evolution equation (18). Limit theorems were developed for multiplicative walks on basis blades and for additive walks thereby induced.

Example 3.9. Letting $(\mathbf{x}_k : 1 \leq k)$ denote a random sequence of basis vectors from $\mathcal{B}$, consider the evolution equation

$$\partial_t u = (\Lambda_{\mathbf{x}_k} + \Xi_{\mathbf{x}_k})u. \tag{21}$$

This equation is associated with a discrete additive process $(u(k))$ in $\mathcal{F}$ associated with a random walk on an infinite-dimensional directed hypercube. In particular, consecutive increments of the process are associated with adjacent vertices in the hypercube. The process $(u(k))$ in $\mathcal{C}\ell_Q(V)$ is of the form

$$u(k) = u(k-1) + (\Lambda_{\mathbf{x}_k} + \Xi_{\mathbf{x}_k})u(k-1). \tag{22}$$

Setting $\Upsilon_\ell := \Lambda_{\mathbf{x}_\ell} + \Xi_{\mathbf{x}_\ell}$ for each ℓ, and using back-substitution, $u(k+1)$ is given by the ordered product

$$u(k) = (\mathcal{I} + \Upsilon_k)(\mathcal{I} + \Upsilon_{k-1}) \cdots (\mathcal{I} + \Upsilon_1)u_0, \tag{23}$$

where $u_0 = u(0)$ is the initial value of the process. Expanding and rewriting with ordered products of lowering and raising operators, one finds

$$u(k) = \sum_{g=0}^{k} \sum_{\substack{(j_1,\ldots,j_g)\in[k]^g \\ j_1>\cdots>j_g}} (\Lambda_{\mathbf{x}_{j_1}} + \Xi_{\mathbf{x}_{j_1}}) \cdots (\Lambda_{\mathbf{x}_{j_g}} + \Xi_{\mathbf{x}_{j_g}})u_0. \tag{24}$$

4 OPERATOR HOMOLOGY & COHOMOLOGY

While a detailed treatment of homology and cohomology is beyond the scope of this paper, an outline of ideas may be helpful.

Definition 4.1. Given a vector space V, vector subspace $W \prec V$, and quadratic form Q defined on V, denote by $\mathcal{C}\ell_Q(W)$ the *subalgebra of* $\mathcal{C}\ell_Q(V)$ *associated with* W by the restriction of Q to W.

Beginning with an order-n, rank-n invertible homogeneous Appell system $\Psi = (\{\psi_k\}, \{\Lambda_{\mathbf{x}_k}\}, \{\Xi_{\mathbf{x}_k}\})$ defined in the Clifford algebra $\mathcal{C}\ell_Q(V)$ associated with an n-dimensional vector space V, the *Appell decomposition of* V *induced by* Ψ is determined as follows: for each $k = 1, \ldots, n-1$, define

$$W_k := \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \tag{25}$$

and set

$$W_0 := \mathbb{R}. \tag{26}$$

In this way, an ascending chain of vector spaces

$$\mathbb{R} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{n-1} \subset W_n = V$$

is obtained. As a consequence, an associated ascending chain of Clifford algebras $\mathcal{C}\ell_Q(W_k)$ satisfying

$$\mathbb{R} \simeq \mathcal{C}\ell_Q(W_0) \subset \mathcal{C}\ell_Q(W_1) \subset \cdots \subset \mathcal{C}\ell_Q(W_n) = \mathcal{C}\ell_Q(V). \tag{27}$$

is induced. Note that for each $k = 0, 1, \ldots, n$,

$$\dim \mathcal{C}\ell_Q(W_k) = 2^k.$$

An important property of the lowering and raising operators is seen in the next lemma.

Lemma 4.2. *For any non-null vector $\mathbf{x} \in \mathcal{C}\ell_Q(V)$, the corresponding generalized lowering and raising operators are nilpotent of index 2. That is,*

$$\Lambda_{\mathbf{x}}{}^2 := \Lambda_{\mathbf{x}} \circ \Lambda_{\mathbf{x}} = 0, \tag{28}$$

$$\Xi_{\mathbf{x}}{}^2 := \Xi_{\mathbf{x}} \circ \Xi_{\mathbf{x}} = 0. \tag{29}$$

An immediate consequence is that solutions of the Clifford *heat equation* must be constant.

Lemma 4.3. *For fixed non-null vector $\mathbf{x}$ in $\mathcal{C}\ell_Q(V)$ and Clifford-valued $u := u(t)$, the Clifford heat equation*

$$\partial_t u = \Lambda_{\mathbf{x}}{}^2 u \tag{30}$$

has only solutions of the form $u(t) = c$ for some constant c.

Several other consequences follow from the standard theory of nilpotent linear operators (cf. [4, Ch. 7]). Each has minimal polynomial $m(t) = t^2$ and characteristic polynomial $\phi(t) = t^{2^n}$. For each, there exists an ordered basis of $\mathcal{C}\ell_Q(V)$ such that the operator's matrix representation with respect to this basis is triangular.

In fact, when basis blades are written in terms of the generating set $\{\mathbf{x}_i : 1 \leq i \leq n\}$, the right regular representation of $\Lambda_{\mathbf{x}_i}$ can be expressed as

$$\Lambda_{\mathbf{x}_i} \mapsto \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ Q(\mathbf{x}_i)\mathcal{E} & \mathbf{0} \end{pmatrix}, \tag{31}$$

where $\mathcal{E}$ is the diagonal $2^{n-1} \times 2^{n-1}$ matrix defined by

$$\mathcal{E}_{I,I} = (-1)^{|I|}. \tag{32}$$

Note that $\mathcal{E}$ acts as grade involution on the subalgebra $\mathcal{C}\ell_Q(\operatorname{span}(\mathbf{x}_i)^\perp)$.

Similarly, the right regular representation of $\Xi_{\mathbf{x}}$ can be expressed as

$$\Xi_{\mathbf{x}_i} \mapsto \begin{pmatrix} \mathbf{0} & \mathcal{E} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{33}$$

It is worth noting that an alternate ordering allows one to replace $\mathcal{E}$ with the identity matrix $\mathcal{I}$ of the same order.

4.1 Cycles and Boundaries

For $1 \leq i \leq n$, the lowering operator $\Lambda_{\mathbf{x}_i} : \mathcal{C}\ell_Q(V) \to \mathcal{C}\ell_Q(V)$ leads to the chain complex

$$\cdots \xrightarrow{\Lambda_{\mathbf{x}_i}} \mathcal{C}\ell_Q(V) \xrightarrow{\Lambda_{\mathbf{x}_i}} \mathcal{C}\ell_Q(V) \xrightarrow{\Lambda_{\mathbf{x}_i}} \cdots. \tag{34}$$

The cycles associated with $\Lambda_{\mathbf{x}_i}$ are the same at each stage and are defined by

$$Z_i = \{u \in \mathcal{C}\ell_Q(V) : \Lambda_{\mathbf{x}_i} u = 0\} = \{u \in \mathcal{C}\ell_Q(V) : \mathbf{x}_i \lrcorner u = 0\}. \tag{35}$$

The boundaries associated with $\Lambda_{\mathbf{x}_i}$ are the same at each stage and are defined by

$$B_i = \{u \in \mathcal{C}\ell_Q(V) : u = \Lambda_{\mathbf{x}_i} w, \text{ for some } w \in \mathcal{C}\ell_Q(V)\} = \{u \in \mathcal{C}\ell_Q(V) : \mathbf{x}_i \lrcorner u = 0\}. \tag{36}$$

In other words, the following condition is satisfied at each stage of the chain complex:

$$\ker(\Lambda_{\mathbf{x}_i}) = \mathrm{Im}(\Lambda_{\mathbf{x}_i}), \tag{37}$$

leading to the trivial homology group $\ker(\Lambda_{\mathbf{x}_i}) / \mathrm{Im}\Lambda_{\mathbf{x}_i} \cong \langle e \rangle$ at each stage.

It follows that the image of $\Lambda_{\mathbf{x}_i}$ is a subalgebra of dimension 2^{n-1} generated by the collection $\{\mathbf{x}_j\}_{j \neq i}$. In particular,

$$\mathcal{C}\ell_Q(V)/\ker(\Lambda_{\mathbf{x}_i}) \cong \mathcal{C}\ell_Q({W_i}^{\perp}). \tag{38}$$

The collection $\{\Lambda_{\mathbf{x}_i}\}_{1 \leq i \leq n}$ then induces the following sequence of epimorphisms:

$$\mathcal{C}\ell_Q(V) \xrightarrow{\Lambda_{\mathbf{x}_n}} \mathcal{C}\ell_Q(W_{n-1}) \xrightarrow{\Lambda_{\mathbf{x}_{n-1}}} \cdots \xrightarrow{\Lambda_{\mathbf{x}_1}} \mathcal{C}\ell_Q(W_0) \simeq \mathbb{R}. \tag{39}$$

Similarly, for $1 \leq i \leq n$, the raising operator $\Xi_{\mathbf{x}_i} : \mathcal{C}\ell_Q(V) \to \mathcal{C}\ell_Q(V)$ leads to the chain complex

$$\cdots \xrightarrow{\Xi_{\mathbf{x}_i}} \mathcal{C}\ell_Q(V) \xrightarrow{\Xi_{\mathbf{x}_i}} \mathcal{C}\ell_Q(V) \xrightarrow{\Xi_{\mathbf{x}_i}} \cdots. \tag{40}$$

The cycles associated with $\Xi_{\mathbf{x}_i}$ are the same at each stage and are defined by

$$Z_i = \{u \in \mathcal{C}\ell_Q(V) : \Xi_{\mathbf{x}_i} u = 0\} = \{u \in \mathcal{C}\ell_Q(V) : \mathbf{x}_i \wedge u \neq 0\}. \tag{41}$$

The boundaries associated with $\Xi_{\mathbf{x}_i}$ are the same at each stage and are defined by

$$B_i = \{u \in \mathcal{C}\ell_Q(V) : u = \Xi_{\mathbf{x}_i} w, \text{ for some } w \in \mathcal{C}\ell_Q(V) = \{u \in \mathcal{C}\ell_Q(V) : \mathbf{x}_i \wedge u \neq 0\}. \tag{42}$$

Hence, the following condition is satisfied at each stage of the chain complex:

$$\ker(\Xi_{\mathbf{x}_i}) = \mathrm{Im}(\Xi_{\mathbf{x}_i}), \tag{43}$$

leading to the trivial homology group $\ker(\Xi_{\mathbf{x}_i}) / \mathrm{Im}(\Xi_{\mathbf{x}_i}) \cong \langle e \rangle$ at each stage.

Considering the algebra isomorphism $\mathcal{C}\ell_Q(V)/\ker(\Xi_{\mathbf{x}_i}) \cong \mathrm{Im}(\Xi_{\mathbf{x}_i})$, it follows that the image of $\Xi_{\mathbf{x}_i}$ is isomorphic to the 2^{n-1}-dimensional subalgebra generated by the collection $\{\mathbf{x}_j\}_{j \neq i}$. In other words,

$$\mathcal{C}\ell_Q(V)/\ker(\Xi_{\mathbf{x}_i}) \cong \mathcal{C}\ell_Q({W_i}^{\perp}). \tag{44}$$

The collection $\{\Xi_{\mathbf{x}_i}\}_{1 \leq i \leq n}$ then induces the following sequence of monomorphisms:

$$\mathbb{R} \simeq \mathcal{C}\ell_Q(W_0) \xrightarrow{\Xi_{\mathbf{x}_1}} \mathcal{C}\ell_Q(W_1) \xrightarrow{\Xi_{\mathbf{x}_2}} \cdots \xrightarrow{\Xi_{\mathbf{x}_n}} \mathcal{C}\ell_Q(V). \tag{45}$$

Much more can be discovered about these chains and their generalizations by considering matrix representations of the lowering and raising operators. Considering sums of lowering and raising operators leads to further insights, including an interpretation of matrix representations as quantum random variables.

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