Nilpotent Adjacency Matrices and Random Graphs

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Abstract

While powers of the adjacency matrix of a finite graph reveal information about walks on the graph, they fail to distinguish closed walks from cycles. Using elements of an appropriate commutative, nilpotent-generated algebra, a "new" adjacency matrix can be associated with a random graph on n vertices and $|E|$ edges of nonzero probability. Letting X_k denote the number of k-cycles occurring in a random graph, this algebra together with a probability mapping allow $\mathbb{E}(X_k)$ to be recovered in terms of $\operatorname{tr} A^k$. Higher moments of X_k can also be computed, and conditions are given for the existence of higher moments in growing sequences of random graphs by considering infinite-dimensional algebras. The algebras used can be embedded in algebras of fermion creation and annihilation operators, establishing connections with quantum computing and quantum probability theory. In the framework of quantum probability, the nilpotent adjacency matrix of a finite graph is a quantum random variable whose m^{th} moment corresponds to the m-cycles contained in the graph.

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1 Introduction

The reader is referred to [8] for essential graph theory terminology and notation. All graphs in this work are assumed to contain no multiple edges and no loops. Graphs may be directed or undirected.

When working with a finite graph G on n vertices, one often utilizes the *adjacency matrix* A associated with G. If the vertices are labeled $\{1, \ldots, n\}$,

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one defines A by

$$A_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

A simple but useful result of this definition, which can also be generalized to directed graphs, is given here without proof.

Proposition 1.1. *Let G be a graph on n vertices with associated adjacency matrix $\mathcal{A}_G$. Then for any positive integer k, the $(i,j)^{th}$ entry of ${\mathcal{A}_G}^k$ is the number of k-walks $i \to j$. In particular, the entries along the main diagonal of ${\mathcal{A}_G}^k$ are the numbers of closed k-walks in G.*

What the adjacency matrix fails to provide, however, is a method of counting self-avoiding walks and cycles in G. This problem is overcome by constructing a nilpotent adjacency matrix.

The methods employed here are original with the authors. The technique involves mapping combinatorial structures into algebras where self-intersections are "sieved out" by multiplication. Then the remaining structures, representing cycles and paths, are recovered by projection.

Other algebraic-probabilistic approaches to graph theory include the works of Hashimoto, Hora, and Obata [2] and Obata [4]. Overlaps between quantum probability and graph theory have also been discussed by Lehner [3].

1.1 Algebraic preliminaries

Let ${\mathcal{C}\ell_n}^{\text{nil}}$ denote the abelian algebra generated by the collection $\{\zeta_i\}$ $(1 \leq i \leq n)$ along with the scalar $1 = \zeta_0$ subject to the following multiplication rules:

$$\zeta_i\,\zeta_j = \zeta_j\,\zeta_i \quad \text{for } i \neq j, \text{ and} \tag{1.2}$$

$$\zeta_i{}^2 = 0 \quad \text{for } 1 \leq i \leq n. \tag{1.3}$$

A general element $\alpha \in {\mathcal{C}\ell_n}^{\text{nil}}$ can be expanded as

$$\alpha = \sum_{\underline{i} \in \mathcal{P}([n])} \alpha_{\underline{i}}\,\zeta_{\underline{i}}\,, \tag{1.4}$$

where $\underline{i} \in \mathcal{P}([n])$ is an element of the power set of $[n] = \{1, 2, \ldots, n\}$ used as a multi-index, $\alpha_{\underline{i}} \in \mathbb{R}$, and $\zeta_{\underline{i}} = \prod_{\iota \in \underline{i}} \zeta_\iota$.

Let ${\mathcal{C}\ell_n}^{\text{idem}}$ denote the abelian algebra generated by the collection $\{\gamma_i\}$ $(1 \leq i \leq n)$ along with the scalar $1 = \gamma_0$ subject to the following multiplication rules:

$$\gamma_i\,\gamma_j = \gamma_j\,\gamma_i \quad \text{for } i \neq j, \text{ and} \tag{1.5}$$

$$\gamma_i{}^2 = \gamma_i \quad \text{for } 1 \leq i \leq n. \tag{1.6}$$

It is evident that a general element $\beta \in {\mathcal{C}\ell_n}^{\text{idem}}$ can also be expanded as in (1.4).

The *inner-product* is defined by

$$\langle u, v\rangle = \left\langle \sum_{\underline{i}\in\mathcal{P}([n])} u_{\underline{i}}\,\zeta_{\underline{i}}\,, \sum_{\underline{j}\in\mathcal{P}([n])} v_{\underline{j}}\,\zeta_{\underline{j}} \right\rangle = \sum_{\underline{i}\in\mathcal{P}([n])} u_{\underline{i}}\,v_{\underline{i}}. \tag{1.7}$$

Hence, arbitrary $u \in \mathcal{C}\ell_n{}^{\mathrm{nil}}$ has the canonical decomposition

$$u = \sum_{\underline{i}\in\mathcal{P}([n])} \left\langle u, \zeta_{\underline{i}}\right\rangle\, \zeta_{\underline{i}}. \tag{1.8}$$

Finally, define the double angle bracket to mean the sum of all scalar coefficients. That is, for $u \in \mathcal{C}\ell_n{}^{\mathrm{nil}}$,

$$\langle\langle u\rangle\rangle = \sum_{\underline{i}\in\mathcal{P}([n])} u_{\underline{i}}\,. \tag{1.9}$$

1.2 Nilpotent Adjacency Matrices

Definition 1.2. Define the *nilpotent adjacency matrix* associated with G by

$$A_{ij} = \begin{cases} \zeta_j, & \text{if } (v_i, v_j) \in E(G) \\ 0, & \text{otherwise.} \end{cases} \tag{1.10}$$

Observe that $A \in \mathrm{Mat}(\mathcal{C}\ell_n{}^{\mathrm{nil}}, n)$, the algebra of $n \times n$ matrices with entries in the abelian nilpotent-generated algebra $\mathcal{C}\ell_n{}^{\mathrm{nil}}$.

Proposition 1.3. *Let A be the nilpotent adjacency matrix of a graph G on n vertices. For any $m > 1$ and $i \neq j$, summing the coefficients of $(A^m)_{ii}$ yields the number of m-cycles based at v_i occurring in G.*

Proof. Proof is by induction on m. When $m = 2$,

$$\left(\mathcal{A}^2\right)_{ii} = (\mathcal{A}\times\mathcal{A})_{ii} = \sum_{\ell=1}^{n} \mathcal{A}_{i\ell}\mathcal{A}_{\ell i}. \tag{1.11}$$

By construction of the nilpotent adjacency matrix,

$$\mathcal{A}_{i\ell} \equiv \text{1-paths } v_i \to v_\ell, \text{ and} \tag{1.12}$$

$$\mathcal{A}_{\ell i} \equiv \text{ 1-path } v_\ell \to v_i. \tag{1.13}$$

Hence, the product of these terms corresponds to 2-cycles $v_i \to v_i$.

Now assuming the proposition holds for m and considering the case $m+1$,

$$\left(\mathcal{A}^{m+1}\right)_{ii} = (\mathcal{A}^m \times \mathcal{A})_{ii} = \sum_{\ell=1}^{n} (\mathcal{A}^m)_{i\ell}\, \mathcal{A}_{\ell i}. \tag{1.14}$$

Considering a general term of the sum,

$$(\mathcal{A}^m)_{i\ell} = \sum_{m\text{-paths } w_m : v_i \to v_\ell} w_m, \text{ and} \tag{1.15}$$

$$\mathcal{A}_{\ell i} = \sum_{1\text{-paths } w_1 : v_\ell \to v_i} w_1. \tag{1.16}$$

It should then be clear that terms of the product

$$(\mathcal{A}^m)_{i\ell}\, \mathcal{A}_{\ell i} \tag{1.17}$$

are nonzero if and only if they correspond to $m{+}1$-paths $v_i \to v_\ell \to v_i$. Summing over all vertices v_ℓ gives the sum of all $m+1$-cycles based at v_i. □

Because A has entries in ${\mathcal{C}\ell_n}^{\text{nil}}$, A^k is identically the zero matrix for all $k > n$. As a result, $(I - tA)^{-1}$ exists as the finite sum $\displaystyle\sum_{k=0}^{n} t^k\, A^k$ for real parameter t, and $\operatorname{tr} A^k$ is recovered as the ${\mathcal{C}\ell_n}^{\text{nil}}$-valued coefficient of t^k in the power series expansion of $\operatorname{tr}(I - tA)^{-1}$.

Example 1.4. The 5-cycles contained in the randomly generated graph in Figure 1.1 are recovered by examining the trace of $\mathcal{A}^5$. Dividing by five compensates for the five choices of base point and dividing by two compensates for possible orientations.

A nilpotent adjacency matrix for random graphs is defined by attaching edge existence probabilities to the nilpotent generators of ${\mathcal{C}\ell_n}^{\text{nil}}$. Using this approach, $\mathbb{E}(X_k)$ is recovered from the trace of A^k [7].

In the number of algebra multiplications required, cycle enumeration is reduced to matrix multiplication. Hence, the time complexity of enumerating a graph's k-cycles requires no more than $O(kn^3)$ algebra multiplications. Several NP-complete problems are moved into class P in this context[6].

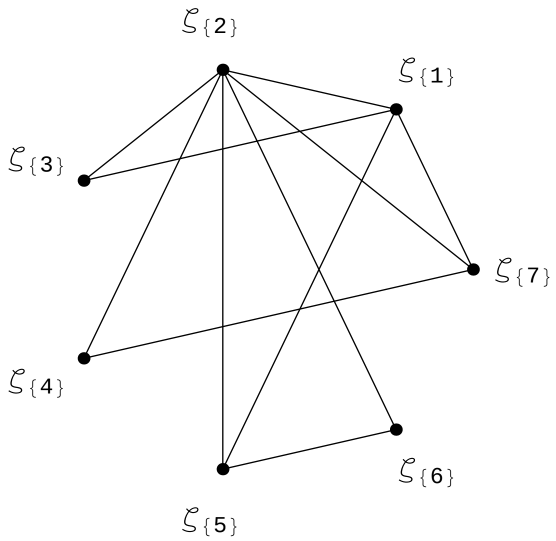
However, computing higher moments of X_k requires computing probabilities $\mathbb{P}(X_k = \ell)$ for $\ell \geq 0$, and the abelian nilpotent-generated algebra ${\mathcal{C}\ell_n}^{\text{nil}}$ is not sufficient for this purpose. In order to compute higher moments, it is necessary to define a nilpotent adjacency matrix with entries in ${\mathcal{C}\ell_n}^{\text{nil}} \otimes {\mathcal{C}\ell_{|E|}}^{\text{idem}}$, where n denotes the number of vertices and $|E|$ denotes the number of edges in the associated graph.

2 Cycles in random graphs

Consider a random graph $G_n = (V_n, E_n)$ on n vertices, $V_n = \{v_1, \ldots, v_n\}$ and $|E_n|$ edges, $E_n = \{(v_{i_1}, v_{j_1}), \ldots, (v_{i_{|E_n|}}, v_{j_{|E_n|}})\}$. Let $2 \leq k \leq n$, and let $\omega \in \{1, 2\}$ be defined by

$$\omega = \begin{cases} 1 & \text{if } G_n \text{ is directed or } k = 2 \\ 2 & \text{otherwise.} \end{cases} \tag{2.1}$$

In[50]:= **NilpotentLabeledPlotGraph[A]**



In[58]:= **NilpotentAdjacencyMatrix[A] // MatrixForm**

Out[58]//MatrixForm=

$$\begin{pmatrix} 0 & \zeta_{\{2\}} & \zeta_{\{3\}} & 0 & \zeta_{\{5\}} & 0 & \zeta_{\{7\}} \\ \zeta_{\{1\}} & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & \zeta_{\{5\}} & \zeta_{\{6\}} & \zeta_{\{7\}} \\ \zeta_{\{1\}} & \zeta_{\{2\}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{\{2\}} & 0 & 0 & 0 & 0 & \zeta_{\{7\}} \\ \zeta_{\{1\}} & \zeta_{\{2\}} & 0 & 0 & 0 & \zeta_{\{6\}} & 0 \\ 0 & \zeta_{\{2\}} & 0 & 0 & \zeta_{\{5\}} & 0 & 0 \\ \zeta_{\{1\}} & \zeta_{\{2\}} & 0 & \zeta_{\{4\}} & 0 & 0 & 0 \end{pmatrix}$$

In[55]:= **Simplify[Tr[ClMatrixPower[M, 5]] / 2 / 5]**

Out[55]= $\zeta_{\{1,2,3,4,7\}} + \zeta_{\{1,2,3,5,6\}} + \zeta_{\{1,2,4,5,7\}} + \zeta_{\{1,2,5,6,7\}}$

In[56]:= **ScalarSum[%]**

Out[56]= 4

Figure 1.1: A randomly generated graph on 7 vertices.

For each ordered pair $(v_i, v_j) \in V(G_n) \times V(G_n)$, define the probability of existence of edge (v_i, v_j) in the graph G_n by

$$p_{ij} = \mathbb{P}\{(v_i, v_j) \in E(G_n)\}. \tag{2.2}$$

Let $\psi : [n] \times [n] \to \left[\frac{2}{\omega}\binom{n}{2}\right]$ be an enumeration of ordered pairs of vertices, excluding the diagonal. Because G_n is assumed to contain no loops, $\psi(i,i)$ is defined to be zero for all $1 \leq i \leq n$.

Defining the random variable X_k as the number of k-cycles occurring in the graph, the goal is to compute $\mathbb{E}(X_k)$ as well as the variance and the higher moments.

Definition 2.1. Labeling the vertices with nilpotents and edges with idempotents, the *nilpotent adjacency matrix of* G_n is defined by

$$A_{i\,j} = \gamma_{\psi(i,j)}\,\zeta_j \in \mathrm{Mat}\left(\mathcal{C}\ell_{|E_n|}{}^{\mathrm{idem}} \otimes \mathcal{C}\ell_n{}^{\mathrm{nil}}, n\right) \tag{2.3}$$

for $1 \leq i, j \leq n$. Here $\mathrm{Mat}\left(\mathcal{C}\ell_{|E_n|}{}^{\mathrm{idem}} \otimes \mathcal{C}\ell_n{}^{\mathrm{nil}}, n\right)$ denotes the algebra of $n \times n$ matrices with entries in $\mathcal{C}\ell_{|E_n|}{}^{\mathrm{idem}} \otimes \mathcal{C}\ell_n{}^{\mathrm{nil}}$.

Definition 2.2. Let $u \in \mathcal{C}\ell_{|E_n|}{}^{\mathrm{idem}} \otimes \mathcal{C}\ell_n{}^{\mathrm{nil}}$ for some $n, |E_n| > 0$, and define

$$\varphi_n = \sum_{\underline{i} \in \mathcal{P}([|E_n|])} \left(\prod_{\iota \in \underline{i}} p_\iota\right) \gamma_{\underline{i}} \in \mathcal{C}\ell_{|E_n|}{}^{\mathrm{idem}}. \tag{2.4}$$

The φ_n*-evaluation* of u is then defined as the nonnegative linear functional

$$\langle\cdot\rangle_{\varphi_n} : \mathcal{C}\ell_{|E_n|}{}^{\mathrm{idem}} \otimes \mathcal{C}\ell_n{}^{\mathrm{nil}} \to \mathbb{R}\,,$$

$$\langle u\rangle_{\varphi_n} = \langle \sum_{\substack{\underline{i} \in \mathcal{P}([|E_n|]) \\ \underline{j} \in \mathcal{P}([n])}} u_{\underline{i}\,\underline{j}}\, \gamma_{\underline{i}}\, \zeta_{\underline{j}}, \varphi_n\rangle = \sum_{\substack{\underline{i} \in \mathcal{P}([n]) \\ \underline{j} \in \mathcal{P}([n])}} |u_{\underline{i}\,\underline{j}}\, \varphi_{n\underline{i}}|\,, \tag{2.5}$$

where $\varphi_{n\underline{i}}$ denotes the product $\prod_{\iota \in \underline{i}} p_\iota$.

If $u = u_{\underline{i}\,\underline{j}}\, \gamma_{\underline{i}}\, \zeta_{\underline{j}}$ for some $\underline{i} \in \mathcal{P}([|E_n|])$, $\underline{j} \in \mathcal{P}([n])$ where $|\underline{i}| = k$, and $|\underline{j}| = \ell$, then u is referred to as a $k \otimes \ell$*-vector.*

When $k \geq 3$, $\mathrm{tr}\, A^k$ will give ωk copies of each k-cycle in G_n. In the particular case $k = 2$, only two copies will be obtained because only one orientation is possible. Let

$$\tau(k,n) = \frac{1}{\omega k} \mathrm{tr}\, A_n{}^k. \tag{2.6}$$

Because the graph contains no multiple edges and no loops, $\tau(1,n) = 0$, and all values of k are hereby assumed to be greater than or equal to 2. Then $\tau(k,n)$ represents a collection of $k \otimes k$-vectors associated with the edges and vertices belonging to the k-cycles of nonzero probability in G_n. Because the edge probabilities are independent, the φ_n-evaluation of each $k \otimes k$-vector is the probability of existence of a k-cycle in G_n. Further,

$$\mathbb{E}(X_k(n)) = \sum_{k\text{-cycles}} \mathbb{P}\{(U_i)\} = \langle \tau(k,n) \rangle_{\varphi_n} , \tag{2.7}$$

where U_i denotes the event that the i^{th} k-cycle exists, $X_k(n)$ is the number of k-cycles in G_n, and $\langle \tau(k,n) \rangle_{\varphi_n}$ denotes the φ_n-evaluation of $\tau(k,n)$.

Now define the map

$$\vartheta_1 : \mathcal{C}\ell_{|E_n|}{}^{\text{idem}} \otimes \mathcal{C}\ell_n{}^{\text{nil}} \to \mathcal{C}\ell_{|E_n|}{}^{\text{idem}} \otimes \mathcal{C}\ell_{2^{|E_n|}}{}^{\text{nil}}$$

by linear extension of

$$\vartheta_1 \left(\gamma_{\underline{\ell}} \, \zeta_{\underline{j}} \right) = \gamma_{\underline{\ell}} \, \zeta_{b_1(\underline{\ell})} , \tag{2.8}$$

where $\underline{\ell} \in \mathcal{P}([|E_n|])$ is a fixed multi-index, $\underline{j} \in \mathcal{P}([n])$ is an arbitrary multi-index, and $b_1 : \mathcal{P}([|E_n|]) \to [2^{|E_n|}]$ is a one-to-one mapping from the power set to the integers $\{1, 2, \ldots, 2^{|E_n|}\}$.

Define the map

$$\vartheta_2 : \mathcal{C}\ell_{|E_n|}{}^{\text{idem}} \otimes \mathcal{C}\ell_{2^{|E_n|}}{}^{\text{nil}} \to \mathcal{C}\ell_{|E_n|}{}^{\text{idem}} \otimes \mathcal{C}\ell_{4^{|E_n|}}{}^{\text{nil}}$$

by linear extension of

$$\vartheta_2 \left(\gamma_{\underline{\ell}} \, \zeta_{\underline{j}} \right) = \gamma_{\underline{\ell}} \, \zeta_{b_2(\underline{j})} , \tag{2.9}$$

where $\underline{\ell} \in \mathcal{P}([|E_n|])$ is a fixed multi-index, $\underline{j} \in 2^{[2^{|E_n|}]}$ is an arbitrary multi-index, and $b_2 : \mathcal{P}\left([2^{|E_n|}]\right) \to [4^{|E_n|}]$ is a one-to-one mapping from the power set to the integers $\{1, 2, \ldots, 4^{|E_n|}\}$.

An easy realization of maps b_1 and b_2 is to think of multi-indices as binary representations of integers. Images of multi-indices under maps b_1 and b_2 are then the integers themselves.

It is worth noting that a graph on n vertices contains at most $\binom{n}{k}\dfrac{(k-1)!}{\omega}$ k-cycles. The maximum number of ℓ-tuples of k-cycles is then $\binom{\binom{n}{k}\frac{(k-1)!}{\omega}}{\ell}$, and the maximum number of j-tuples of ℓ-tuples of k-cycles is given by $\binom{\binom{\binom{n}{k}\frac{(k-1)!}{\omega}}{\ell}}{j}$. Where these quantities appear, they should be regarded in this context.

Proposition 2.3.

$$\mathbb{P}\{X_k(n) = \ell\} = \left\langle \frac{1}{(\ell+1)!} \exp\left(\frac{-\vartheta_2 \left(\vartheta_1 \left(\mathrm{tr} {A_n}^k\right)^{\ell+1}\right)}{(\omega\, k)^{\ell+1}} \right) \right\rangle_{\varphi_n} - \left\langle \frac{1}{\ell!} \exp\left(\frac{-\vartheta_2 \left(\vartheta_1 \left(\mathrm{tr} {A_n}^k\right)^{\ell}\right)}{(\omega\, k)^{\ell}} \right) \right\rangle_{\varphi_n}. \tag{2.10}$$

Proof. Utilizing idempotency of the edges and nilpotency of the vertices, assuming lexicographical ordering of multi-indices, and expanding $\tau(k,n)$ in terms of the k-cycles it represents, $\tau(k,n) = \sum_{i=1}^{\binom{n}{k}\frac{(k-1)!}{\omega}} \tau(k,n)_i$, one can see that

$$\sum_{i<j} \tau(k,n)_i\, \tau(k,n)_j$$

gives the collection of $2k \otimes 2k$-vectors associated with edge- and vertex-sets of *pairs* of k-cycles. It is further evident that

$$\sum_{i_1<i_2<\cdots<i_m} \mathbb{P}(U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_m}) = \left\langle \sum_{i_1<\cdots<i_m} \tau(k,n)_{i_1} \cdots \tau(k,n)_{i_m} \right\rangle_{\varphi_n}. \tag{2.11}$$

So the probability that G_n contains one or more k-cycles is

$$\begin{aligned}
\mathbb{P}(U_1 \cup \cdots \cup U_{\binom{n}{k}\frac{(k-1)!}{\omega}}) &= \sum_i \mathbb{P}(U_i) - \sum_{i_1<i_2} \mathbb{P}(U_{i_1} \cap U_{i_2}) \\
&+ \sum_{i_1<i_2<i_3} \mathbb{P}(U_{i_1} \cap U_{i_2} \cap U_{i_3}) - \cdots + (-1)^{[\binom{n}{k}\frac{(k-1)!}{\omega}-1]} U_{i_1} \cap \cdots \cap U_{i_{\binom{n}{k}\frac{(k-1)!}{\omega}}} \\
&= \left\langle \sum_i \tau(k,n)_i \right\rangle_{\varphi_n} - \left\langle \sum_{i<j} \tau(k,n)_i\, \tau(k,n)_j \right\rangle_{\varphi_n} \\
&+ \left\langle \sum_{i<j<\ell} \tau(k,n)_i\, \tau(k,n)_j\, \tau(k,n)_\ell \right\rangle_{\varphi_n} - \cdots \\
&\cdots + (-1)^{[\binom{n}{k}\frac{(k-1)!}{\omega}-1]} \left\langle \tau(k,n)_1 \cdots \tau(k,n)_{\binom{n}{k}\frac{(k-1)!}{\omega}} \right\rangle_{\varphi_n}.
\end{aligned} \tag{2.12}$$

Similarly, the probability that G_n contains *two* or more k-cycles is computed by defining U_i as the event the i^{th} *pair* of k-cycles exists.

Let $\tau(k,n)_i^{(\ell)}$ denote the multivector representation of the edge-set associated with the i^{th} ℓ-tuple of k-cycles occurring in G_n, and assume lexicographical ordering of the multivector indices. In other words,

$$\tau(k,n)^{(\ell)} = \sum_{1 \le i_1 < i_2 < \cdots < i_\ell \le \binom{n}{k}\frac{(k-1)!}{\omega}} \tau(k,n)_{i_1} \cdots \tau(k,n)_{i_\ell}. \tag{2.13}$$

It is now evident that if $X_k(n)$ denotes the number of k-cycles appearing in G_n, the probability that G_n contains ℓ or more k-cycles is equal to the probability one or more ℓ-tuples of k-cycles exist in G_n. In other words,

$$\begin{aligned}\mathbb{P}\{X_k(n) \ge \ell\} = &\sum_{i=1}^{\binom{\binom{n}{k}\frac{(k-1)!}{\omega}}{\ell}} \left\langle \tau(k,n)_i^{(\ell)} \right\rangle_{\varphi_n} - \left\langle \sum_{i<j} \tau(k,n)_i^{(\ell)}\, \tau(k,n)_j^{(\ell)} \right\rangle_{\varphi_n} + \cdots \\ &+ (-1)^{\left[\binom{\binom{n}{k}\frac{(k-1)!}{\omega}}{\ell}-1\right]} \left\langle \tau(k,n)_1^{(\ell)} \cdots \tau(k,n)^{(\ell)}_{\binom{\binom{n}{k}\frac{(k-1)!}{\omega}}{\ell}} \right\rangle_{\varphi_n}. \end{aligned} \tag{2.14}$$

Using nilpotency of multivectors associated with vertices in G_n, one finds

$$\begin{aligned}\tau(k,n)^{(\ell)} = \sum_{i_1 < i_2 < \cdots i_\ell} \tau(k,n)_{i_1} \cdots \tau(k,n)_{i_\ell} &= \frac{1}{\ell!} \vartheta_1(\tau(k,n))^\ell \\ &= \frac{1}{\ell!(\omega\, k)^\ell} \vartheta_1(\mathrm{tr}\, {A_n}^k)^\ell. \end{aligned} \tag{2.15}$$

Similarly,

$$\begin{aligned}\sum_{i_1 < i_2 < \cdots i_j} \tau(k,n)_{i_1}^{(\ell)} \cdots \tau(k,n)_{i_j}^{(\ell)} &= \frac{1}{\ell! j!} \vartheta_2(\vartheta_1(\tau(k,n))^\ell)^j \\ &= \frac{1}{\ell! j! (\omega\, k)^{\ell j}} \vartheta_2(\vartheta_1(\mathrm{tr}\, {A_n}^k)^\ell)^j. \end{aligned} \tag{2.16}$$

Therefore,

$$\mathbb{P}\{X_k(n) = \ell\} = \mathbb{P}\{X_k(n) \ge \ell\} - \mathbb{P}\{X_k(n) \ge \ell+1\}, \tag{2.17}$$

where

$$\begin{aligned}\mathbb{P}\{X_k(n) \ge \ell\} = &\left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^\ell\right)}{\ell!(\omega k)^\ell} \right\rangle_{\varphi_n} - \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^\ell\right)^2}{2\ell!(\omega k)^{2\ell}} \right\rangle_{\varphi_n} + \cdots \\ \cdots + &\frac{(-1)^{\binom{\binom{n}{k}(k-1)!}{\ell}}}{\ell! \binom{\binom{n}{k}(k-1)!}{\ell}! (\omega k)^{\ell \binom{\binom{n}{k}(k-1)!}{\ell}}} \left\langle \vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^\ell\right)^{\binom{\binom{n}{k}(k-1)!}{\ell}} \right\rangle_{\varphi_n}, \end{aligned} \tag{2.18}$$

and

$$\mathbb{P}\{X_k(n) \geq \ell{+}1\} = \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr}{A_n}^k\right)^{\ell+1}\right)}{(\ell+1)!(\omega k)^{\ell+1}} \right\rangle_{\varphi_n} - \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr}{A_n}^k\right)^{\ell+1}\right)^2}{2(\ell+1)!(\omega k)^{2(\ell+1)}} \right\rangle_{\varphi_n} + \cdots$$

$$\cdots + \frac{(-1)^{\binom{\binom{n}{k}(k-1)!}{\ell+1}}}{(\ell+1)!\binom{\binom{n}{k}(k-1)!}{\ell+1}!(\omega k)^{(\ell+1)\binom{\binom{n}{k}(k-1)!}{\ell+1}}} \left\langle \vartheta_2\left(\vartheta_1\left(\mathrm{tr}{A_n}^k\right)^{\ell+1}\right)^{\binom{\binom{n}{k}(k-1)!}{\ell+1}} \right\rangle_{\varphi_n}. \tag{2.19}$$

Hence,

$$\mathbb{P}\{X_k(n) = \ell\} = \sum_{j=1}^{\binom{\binom{n}{k}(k-1)!}{\ell}} (-1)^{j-1} \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr}{A_n}^k\right)^{\ell}\right)^j}{\ell! j! (\omega k)^{j\ell}} \right\rangle_{\varphi_n} - \sum_{j=1}^{\binom{\binom{n}{k}(k-1)!}{\ell+1}} (-1)^{j-1} \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr}{A_n}^k\right)^{\ell+1}\right)^j}{(\ell+1)! j! (\omega k)^{j(\ell+1)}} \right\rangle_{\varphi_n}. \tag{2.20}$$

Observing that $\vartheta_2\left(\vartheta_1\left(\mathrm{tr}{A_n}^k\right)^{\ell}\right)^j = 0$ whenever $k > n$, $\ell > \binom{n}{k}(k-1)!$ or $j > \binom{\binom{n}{k}(k-1)!}{\ell}$, one has

$$\mathbb{P}\{X_k(n) = \ell\} = \sum_{j=0}^{\infty} \frac{(-1)^{j-1}}{\ell! j!} \left(\left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr}{A_n}^k\right)^{\ell}\right)^j}{(\omega\, k)^{\ell j}} \right\rangle_{\varphi_n} - \frac{1}{\ell+1} \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr}{A_n}^k\right)^{\ell+1}\right)^j}{(\omega k)^{j(\ell+1)}} \right\rangle_{\varphi_n} \right). \tag{2.21}$$

Rewriting the infinite series gives

$$\mathbb{P}\{X_k(n) = \ell\} =$$

$$\sum_{j=0}^{\infty} \frac{(-1)^{j-1}}{\ell! j!} \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell}\right)^j}{(\omega\, k)^{\ell j}} \right\rangle_{\varphi_n} - \sum_{j=0}^{\infty} \frac{(-1)^{j-1}}{(\ell+1)! j!} \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell+1}\right)^j}{(\omega k)^{j(\ell+1)}} \right\rangle_{\varphi_n}$$

$$= -\sum_{j=0}^{\infty} \frac{(-1)^{j}}{\ell! j!} \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell}\right)^j}{(\omega\, k)^{\ell j}} \right\rangle_{\varphi_n} + \sum_{j=0}^{\infty} \frac{(-1)^{j}}{(\ell+1)! j!} \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell+1}\right)^j}{(\omega k)^{j(\ell+1)}} \right\rangle_{\varphi_n}$$

$$= \left\langle -\frac{1}{\ell!} \exp\left(\frac{-\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell}\right)}{(\omega\, k)^{\ell}}\right) \right\rangle_{\varphi_n} + \left\langle \frac{1}{(\ell+1)!} \exp\left(\frac{-\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell+1}\right)}{(\omega\, k)^{\ell+1}}\right) \right\rangle_{\varphi_n}$$

$$= \left\langle \frac{1}{(\ell+1)!} \exp\left(\frac{-\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell+1}\right)}{(\omega\, k)^{\ell+1}}\right) \right\rangle_{\varphi_n} - \left\langle \frac{1}{\ell!} \exp\left(\frac{-\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell}\right)}{(\omega\, k)^{\ell}}\right) \right\rangle_{\varphi_n}. \tag{2.22}$$

□

Corollary 2.4. *Let $n, m > 0$ be fixed and let G_n be a random graph on n-vertices with associated nilpotent adjacency matrix A_n. Then for $k \leq n$,*

$$\mathbb{E}\left(X_k(n)^m\right) = \sum_{\ell=1}^{\infty} \frac{\ell^m}{(\ell+1)!} \left\langle \exp\left(\frac{-\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell+1}\right)}{(\omega\, k)^{\ell+1}}\right) \right\rangle_{\varphi_n} - \sum_{\ell=1}^{\infty} \frac{\ell^m}{\ell!} \left\langle \exp\left(\frac{-\vartheta_2\left(\vartheta_1\left(\mathrm{tr} {A_n}^k\right)^{\ell}\right)}{(\omega\, k)^{\ell}}\right) \right\rangle_{\varphi_n}. \tag{2.23}$$

The variance of $X_k(n)$ is then given by

$$\begin{aligned}
\mathrm{var} X_k(n) &= \mathbb{E}(X_k(n)^2) - \mathbb{E}(X_k(n))^2 \\
&= \sum_{\ell=1}^{\infty} \frac{\ell^2}{(\ell+1)!} \left\langle \exp\left(\frac{-\vartheta_2 \left(\vartheta_1 \left(\mathrm{tr} {A_n}^k \right)^{\ell+1} \right)}{(\omega\, k)^{\ell+1}} \right) \right\rangle_{\varphi_n} \\
&\quad - \sum_{\ell=1}^{\infty} \frac{\ell^2}{\ell!} \left\langle \exp\left(\frac{-\vartheta_2 \left(\vartheta_1 \left(\mathrm{tr} {A_n}^k \right)^{\ell} \right)}{(\omega\, k)^{\ell}} \right) \right\rangle_{\varphi_n} \\
&\quad - \left(\left\langle \frac{1}{\omega k} \mathrm{tr} {A_n}^k \right\rangle_{\varphi_n} \right)^2 .
\end{aligned} \tag{2.24}$$

3 Convergence of Moments

Let $\mathcal{G} = \{G_n\}$ denote a family of random graphs. For each $n > 0$, let G_n denote a random graph on n vertices having $|E_n| = \frac{2}{\omega}\binom{n}{2}$ edges of probability $\{p_1, \ldots, p_{2\binom{n}{2}}\}$. Further assume that each G_n is a subgraph of G_{n+1}. In other words,

$$\begin{aligned}
v_i \in V_n &\Rightarrow v_i \in V_{n+1} \text{ , and} \\
(v_i, v_j) \in V_n \times V_n &\Rightarrow (v_i, v_j) \in V_{n+1} \times V_{n+1}.
\end{aligned}$$

It is apparent that V_{n+1} contains vertex v_{n+1}, and that $V_{n+1} \times V_{n+1}$ contains a collection of edges $\{(v_i, v_{n+1})\}$ and $\{(v_{n+1}, v_i)\}$ where $1 \le i \le n$.

For each $n > 0$, the adjacency matrix of G_n has entries in ${\mathcal{C}\ell_{|E_n|}}^{\mathrm{idem}} \otimes {\mathcal{C}\ell_n}^{\mathrm{nil}}$. Each algebra is therefore canonically embedded in the infinite-dimensional algebra $\mathcal{C}\ell^{\mathrm{idem}} \otimes \mathcal{C}\ell^{\mathrm{nil}}$, defined by

$$\mathcal{C}\ell^{\mathrm{idem}} \otimes \mathcal{C}\ell^{\mathrm{nil}} = \bigoplus_{n=1}^{\infty} \left({\mathcal{C}\ell_{\frac{2}{\omega}\binom{n}{2}}}^{\mathrm{idem}} \otimes {\mathcal{C}\ell_n}^{\mathrm{nil}} \right). \tag{3.1}$$

For each n, let $\varphi_n \in {\mathcal{C}\ell_{|E_n|}}^{\mathrm{idem}}$ be defined as in (2.4). Because G_n is a subgraph of G_{n+1} for all n,

$$\langle u_m \rangle_{\varphi_m} = \langle u_n \rangle_{\varphi_n} \tag{3.2}$$

holds for all $n > m$ whenever $u_m \in {\mathcal{C}\ell_{E_m}}^{\mathrm{idem}}$.

It is required that

$$\varphi = \lim_{n\to\infty} \varphi_n \in \mathcal{C}\ell^{\mathrm{idem}} \otimes \mathcal{C}\ell^{\mathrm{nil}}$$

exists. A necessary and sufficient condition for existence of φ is

$$\|\varphi\|^2 = \lim_{n\to\infty} \|\varphi_n\|^2 = \lim_{n\to\infty} \sum_{\underline{i}\in\mathcal{P}([|E_n|])} |\varphi_{n\underline{i}}|^2 = \lim_{n\to\infty} \sum_{\underline{i}\in\mathcal{P}([|E_n|])} \left(\prod_{\iota\in\underline{i}} p_\iota^2\right) < \infty. \tag{3.3}$$

Note the use of an inner-product norm for the idempotent-generated algebra. The definition of this norm should be clear from the canonical expansion

$$\varphi_n = \sum_{\underline{i}\in\mathcal{P}([|E_n|])} \varphi_{n\,\underline{i}}\,\gamma_{\underline{i}}\,, \tag{3.4}$$

where $\varphi_{n\,\underline{i}} \in \mathbb{R}$ are real scalar coefficients.

Theorem 3.1. *Let $\{G_n\}$ be an increasing sequence of random graphs such that (3.2) and (3.3) are satisfied. Let $k \geq 2$ and $m \geq 1$ be fixed. For each $n \in \mathbb{N}$, let $|V(G_n)| = n$ and let A_n denote the nilpotent adjacency matrix for G_n. Let B_m denote the m^{th} Bell number. Suppose that $\forall\varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $\forall j, \ell \geq 0$ and $\forall n_1, n_2 > N_\varepsilon$ the following inequality is satisfied:*

$$\left|\left\langle \vartheta_2\left(\vartheta_1\left(\operatorname{tr} {A_{n_1}}^k\right)^\ell\right)^j\right\rangle_\varphi - \left\langle \vartheta_2\left(\vartheta_1\left(\operatorname{tr} {A_{n_2}}^k\right)^\ell\right)^j\right\rangle_\varphi\right| \leq \frac{\varepsilon}{eB_m}\,. \tag{3.5}$$

Then $\lim\limits_{n\to\infty} \mathbb{E}(X_k(n)^m)$ exists.

Proof. For fixed ℓ,

$$\sum_j \frac{(-1)^{j-1}\ell^m}{j!\ell!(\omega k)^{\ell j}} = -\frac{\ell^m}{\ell!} e^{-\frac{1}{(\omega k)^\ell}}\,. \tag{3.6}$$

Now $e^{-\frac{1}{(\omega k)^\ell}} < 1\ \forall \omega, k, \ell$, and by Dobinski's Formula [1],

$$\sum_{\ell=0}^{\infty} -\frac{\ell^m}{\ell!} = -eB_m. \tag{3.7}$$

Thus,

$$\left|\sum_{j,\ell} \frac{(-1)^{j-1}\ell^m}{j!\ell!(\omega k)^{\ell j}}\right| \leq \sum_{j,\ell} \frac{\ell^m}{j!\ell!(\omega k)^{\ell j}} = \sum_\ell \frac{\ell^m}{\ell!} e^{-\frac{1}{(\omega k)^\ell}} \leq eB_m. \tag{3.8}$$

Now let $\varepsilon > 0$ be arbitrary and suppose $\exists N_\varepsilon \in \mathbb{N}$ such that $n_1, n_2 > N_\varepsilon$ implies

$$\left|\left\langle \vartheta_2\left(\vartheta_1\left(\operatorname{tr} {A_{n_1}}^k\right)^\ell\right)^j\right\rangle_\varphi - \left\langle \vartheta_2\left(\vartheta_1\left(\operatorname{tr} {A_{n_2}}^k\right)^\ell\right)^j\right\rangle_\varphi\right| \leq \frac{\varepsilon}{eB_m} \tag{3.9}$$

for all $j, \ell \geq 0$. Then

$$\left|\sum_{\ell,j} \frac{(-1)^{j-1}\ell^m}{j!\ell!(\omega k)^{\ell j}} \left(\left\langle \vartheta_2\left(\vartheta_1\left(\operatorname{tr} A_{n_1}{}^k\right)^\ell\right)^j\right\rangle_\varphi - \left\langle \vartheta_2\left(\vartheta_1\left(\operatorname{tr} A_{n_2}{}^k\right)^\ell\right)^j\right\rangle_\varphi\right)\right|$$
$$\leq \sum_{\ell,j} \left|\frac{(-1)^{j-1}\ell^m}{j!\ell!(\omega k)^{\ell j}} \left(\left\langle \vartheta_2\left(\vartheta_1\left(\operatorname{tr} A_{n_1}{}^k\right)^\ell\right)^j\right\rangle_\varphi - \left\langle \vartheta_2\left(\vartheta_1\left(\operatorname{tr} A_{n_2}{}^k\right)^\ell\right)^j\right\rangle_\varphi\right)\right|$$
$$\leq \frac{\varepsilon}{eB_m} \sum_{j,\ell} \left|\frac{(-1)^{j-1}\ell^m}{j!\ell!(\omega k)^{\ell j}}\right| \leq \frac{\varepsilon}{eB_m} eB_m = \varepsilon. \quad (3.10)$$

Thus, $\{\mathbb{E}\left(X_k(n)^m\right)\}$ is a Cauchy sequence, and $\lim_{n\to\infty} \mathbb{E}(X_k(n)^m)$ exists. □

If the m^{th} moment of X_k exists,

$$\mathbb{E}({X_k}^m) =$$
$$\lim_{n\to\infty} \sum_{\ell,j} \frac{(-1)^{j-1}\ell^m}{j!\ell!} \left(\left\langle \frac{\vartheta_2\left(\vartheta_1\left(\operatorname{tr} A_n{}^k\right)^\ell\right)^j}{(\omega k)^{\ell j}}\right\rangle_{\varphi_n} - \left\langle \frac{\vartheta_2\left(\vartheta_1\left(\operatorname{tr} A_n{}^k\right)^{\ell+1}\right)^j}{(\ell+1)(\omega k)^{j(\ell+1)}}\right\rangle_{\varphi_n}\right). \quad (3.11)$$

Moreover,

$$\mathbb{E}(X_k) = \lim_{n\to\infty} \langle \frac{1}{\omega k} \operatorname{tr} A_n{}^k \rangle_{\varphi_n}, \quad (3.12)$$

provided the limit exists. In light of these equations, one has

$$\operatorname{var} X_k = \mathbb{E}({X_k}^2) - \mathbb{E}(X_k)^2, \quad (3.13)$$

provided the limits (3.12) and (3.11) exist for $m = 2$.

Remark 3.2. Because m is fixed, the quantity eB_m could be absorbed into the ε of inequality (3.5). As stated, the theorem reveals a connection between the m^{th} Bell number and the existence of the m^{th} moment.

3.1 Characterizing the Moments

Proposition 3.3. *Let $G = (V, E)$ be a random graph on n vertices. Fix $k \geq 2$, and define the following quantities:*

$$c_\ell = \mathbb{P}\{X_k = \ell\} \quad (3.14)$$
$$\gamma = \max_{c_\ell \neq 0} \ell. \quad (3.15)$$

Then

$$\lim_{m\to\infty} \frac{\mathbb{E}({X_k}^m)}{\gamma^m} = c_\gamma. \quad (3.16)$$

Proof. By definition of the m^{th} moment of X_k,

$$\mathbb{E}({X_k}^m) = \sum_{\ell} \ell^m \, \mathbb{P}\{X_k = \ell\} = \sum_{\ell} \ell^m c_\ell \, . \tag{3.17}$$

Let γ denote the maximum value of ℓ such that $c_\ell \neq 0$. Then

$$\frac{\mathbb{E}({X_k}^m)}{\gamma^m} = \sum_{\ell} \frac{\ell^m}{\gamma^m} c_\ell = c_\gamma + \sum_{\ell \neq \gamma} \frac{\ell^m}{\gamma^m} c_\ell \, . \tag{3.18}$$

Observing that $\ell < \gamma$ and $0 \leq c_\ell \leq 1$, the proof is complete. □

4 Links to Quantum Computing

The algebras ${\mathcal{C}\ell_n}^{\text{nil}}$ and ${\mathcal{C}\ell_n}^{\text{idem}}$ can be constructed within fermion creator / annihilator algebras of appropriate dimension.

The algebra ${\mathcal{C}\ell_n}^{\text{nil}}$ can be constructed within the $2n$-particle fermion algebra by writing

$$\zeta_i = {f_i}^+ f_{n+i}^+. \tag{4.1}$$

Here, ${f_i}^+$ denotes the i^{th} fermion creation operator.

The algebra ${\mathcal{C}\ell_n}^{\text{idem}}$ can also be constructed within the $2n$-particle fermion algebra. Fix $n > 0$ and consider elements of the form

$$\gamma_i = \frac{1}{2}\left(1 + \left(\frac{f_i + {f_i}^+}{2}\right)\left(\frac{f_{n+i} + f_{n+i}^+}{2}\right)\right). \tag{4.2}$$

Here, ${f_i}^+$ denotes the i^{th} fermion creation operator, and f_i denotes the i^{th} fermion annihilation operator.

Direct calculation shows

$$\begin{aligned} {\gamma_i}^2 &= \left(\frac{1 + \left(\frac{f_i + {f_i}^+}{2}\right)\left(\frac{f_{n+i} + f_{n+i}^+}{2}\right)}{2}\right)^2 \\ &= \frac{1}{4} + \frac{1}{2}\left(\frac{f_i + {f_i}^+}{2}\right)\left(\frac{f_{n+i} + f_{n+i}^+}{2}\right) \\ &\quad + \frac{1}{2}\left(\frac{f_i + {f_i}^+}{2}\right)\left(\frac{f_{n+i} + f_{n+i}^+}{2}\right)\left(\frac{f_i + {f_i}^+}{2}\right)\left(\frac{f_{n+i} + f_{n+i}^+}{2}\right) \\ &= \frac{2\left(\frac{f_i + {f_i}^+}{2}\right)\left(\frac{f_{n+i} + f_{n+i}^+}{2}\right) + 2}{4} = \gamma_i. \end{aligned} \tag{4.3}$$

Because each γ_i is written using a pair $(i, n+i)$ of fermion creation/annihilation operator pairs and because these pairs are disjoint for $i \neq j$, direct calculation also shows that $\gamma_i \, \gamma_j = \gamma_j \, \gamma_i$ for $i \neq j$.

Letting $\mathcal{F}$ denote the infinite-dimensional fermion algebra,

$$\mathcal{C}\ell^{\text{idem}} \otimes \mathcal{C}\ell^{\text{nil}} \subset \mathcal{F} \otimes \mathcal{F}.$$

The nilpotent adjacency matrix associated with a finite graph can itself be considered a quantum random variable whose m^{th} moment corresponds to the number of m-cycles occurring in the graph [5]. Considering sequences of such quantum random variables associated with ascending sequences of random graphs is a topic for further research.

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