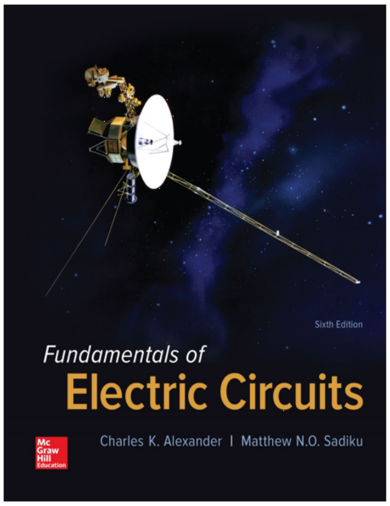


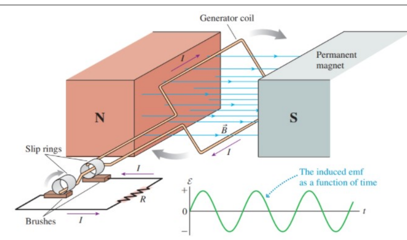
Chapter 9 - Sinusoids and Phasors

- **This chapter will cover alternating current.**
- **A discussion of complex numbers is included prior to introducing phasors.**
- **Applications of phasors and frequency domain analysis for circuits including resistors, capacitors, and inductors will be covered.**
- **The concept of impedance and admittance is also introduced.**

9.1 Introduction

An electrical generator or dynamo produces an electric current through the rotation of a wire coil in a magnetic field. The steady rotation of the coil causes the emf and the induced current in the coil to oscillate sinusoidally, alternating positive and then negative, and the current actually reverses direction with every rotation of the coil. Current that reverses direction in this fashion is known as *alternating current* (AC). PhET "Generator" w/ pickup coil simulation?

A battery, on the other hand, produces an emf through chemical changes within the battery. Since these changes happen at a constant rate, there is no variation in the direction of the emf or of the current, which flows in only one direction, hence the name *direct current* (DC).



As modern electrified infrastructures started to be built a major debate arose about whether the electrical supply system in the United States should be AC or DC. Thomas Edison favored a DC system while AC was supported by the inventor Nikola Tesla and a young entrepreneur by the name of George Westinghouse. Following what is sometimes known as the "War of Currents," a system based on AC eventually won out primarily because it is more efficient to transmit over long distances.

While DC power is not used generally for the transmission of energy from power plants into homes, it is still common when distances are small. It is also widely used in all modern electronic devices, telephones, and automotive systems.

Interestingly, the electrical distribution system in Europe is based on DC. The power supply in most European homes is 230 volts @50Hz while for the US it is 120 volts @ 60Hz.

9.2 Sinusoids

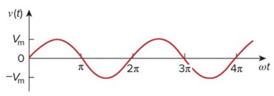
Sinusoids are of interest in many areas because there are a number of natural phenomenon that are sinusoidal in nature. It is also a very easy signal to generate and transmit and, through Fourier analysis, any practical periodic function can be made by adding sinusoids. Finally, they are very easy to handle mathematically.

A sinusoid is a signal that has the form of a sine or cosine function.

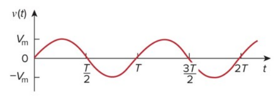
Consider a sinusoidal voltage: $v(t) = V_m \sin \omega t$ where V_m = the amplitude of the sinusoid

ω = the angular frequency in radians/sec

ωt = the argument of the sinusoid



v(t) as a function of the argument



v(t) as a function of time

The sinusoid repeats every T seconds, which is known as the period of the sinusoid. In addition,

$$\omega T = 2\pi \qquad T = \frac{2\pi}{\omega}$$

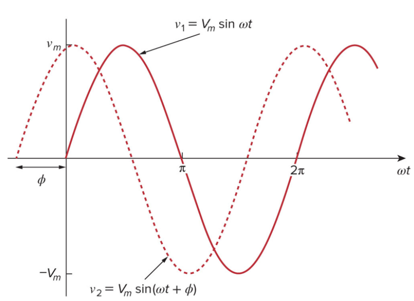
The period is inversely related to another important characteristic, the cyclic frequency (most often simply called *frequency*):

$$f = \frac{1}{T} \quad \text{and} \quad \omega = 2\pi f$$

where f is in hertz (Hz) and ω is in rad/s

If multiple sinusoids (aka waves) are involved, it becomes necessary to account for the relative timing of one versus another. This is done by including a *phase shift*

This can be done by including a phase shift, ϕ .



If two sinusoids are in phase, they reach their maximum and minimum at the same time.

Sinusoids may be expressed either as sine or cosine functions, and the conversions between the functions are given by:

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$

$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$

$$\sin(\omega t \pm 90^\circ) = \pm\cos \omega t$$

$$\cos(\omega t \pm 90^\circ) = \sin \omega t$$

When comparing two sinusoids, it is best to express both as either sine or cosine with positive amplitudes using the identities shown above.

Examples and Practice Problems

E9.1 Find the amplitude, phase, period, angular frequency, and (cyclic) frequency of the sinusoid:

$v(t) = 12 \cos (50t + 10^{\circ})$ V

P9.1 Given the sinusoid $45 \cos (5\pi t + 36^{\circ})$, find the amplitude, phase, angular frequency, period, and (cyclic) frequency.

E9.2 Calculate the phase angle between $v_1 = -10 \cos (\omega t + 50^{\circ})$ and $v_2 = 12 \sin (\omega t - 10^{\circ})$. State which sinusoid is leading.

P9.2 Find the phase angle between $i_1 = -4 \sin (377t + 55^{\circ})$ and $i_2 = 5 \cos (377t - 65^{\circ})$. Does i_1 lead or lag i_2?

9.3 Phasors

A *phasor* is a complex number that represents the amplitude and phase of a sinusoid, are more convenient to work with than sine and cosine functions, and can provide a simple means of analyzing linear circuits excited by sinusoidal sources.

Before looking at phasors, however, a look at complex numbers is necessary.

A complex number ***z*** can be represented in rectangular form as: $z = x + jy$ where $j = \sqrt{-1}$

z can also be written in polar form: $z = r\angle\Phi$

or

exponential form: $z = re^{j\Phi}$

where *r* is the magnitude of ***z*** and Φ is the phase of ***z***.

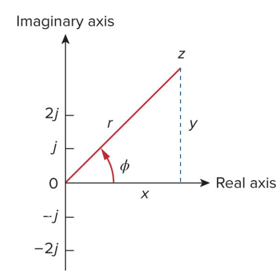
The different forms are interconnected and can be inter-converted.

Starting with rectangular form, one can go to polar:

$$r = \sqrt{x^2 + y^2} \qquad \Phi = \tan^{-1}\frac{y}{x}$$

Likewise, from polar form to rectangular goes as follows:

$$x = r\cos\Phi \qquad y = r\sin\Phi$$



Combining the possible forms: *z* may be written as: $z = x + jy = r\angle\Phi = r(\cos\Phi + j\sin\Phi)$

The usual mathematical operations can be performed with complex numbers although:

addition and subtraction are better performed in rectangular form.

multiplication and division are better done in polar form.

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Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r}\angle(-\phi)$$

Square Root

$$\sqrt{z} = \sqrt{r}\angle(\phi/2)$$

Complex Conjugate

$$z^* = x - jy = r\angle - \phi = re^{-j\phi}$$

Reciprocal of *j*

$$\frac{1}{j} = -j$$

Ch 9, p9

The idea of a phasor representation is based on Euler's identity: $e^{\pm j\Phi} = \cos\Phi \pm j\sin\Phi$

where the first term can be viewed as the real part of $e^{j\Phi}$ and the second as the imaginary part.

From this we can represent a sinusoid as the real component of a vector in the complex plane.

The length of the vector is the amplitude of the sinusoid.

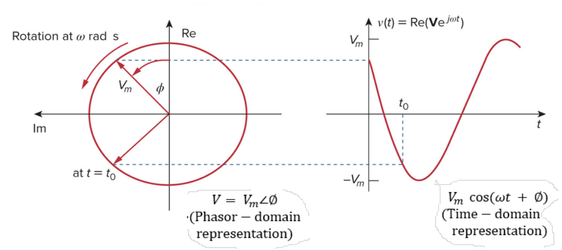
The vector, ***V***, in polar form, is at an angle Φ with respect to the positive real axis.

A sinusoidal voltage *v(t)* can be represented as:

$v(t) = V_m \cos(\omega t + \Phi) = \mathrm{Re}(V_m e^{j(\omega t + \Phi)}) = \mathrm{Re}(V_m e^{j\Phi} e^{j\omega t}) = \mathrm{Re}(\mathbf{V} e^{j\omega t})$ where $\mathbf{V} = V_m e^{j\Phi} = V_m \angle \Phi$

To get the phasor corresponding to a sinusoid:

- express the sinusoid in cosine form so the sinusoid can be written as the real part of a complex number.
- remove the time factor $e^{j\omega t}$.
  - this transforms the sinusoid from the time domain to the phasor domain.
- whatever is left is the phasor corresponding to the sinusoid.



Ch 9, p10

Here is a handy table for transforming various time domain sinusoids into phasor domain:

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90°$
$I_m \cos(\omega t + \phi)$	$I_m \angle \phi$
$I_m \sin(\omega t + \phi)$	$I_m \angle \phi - 90°$

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The standard convention is to use the cosine form to develop the phasor representation and to return to the cosine form from the phasor representation.

$\sin(\omega t \pm 180^\circ) = -\sin \omega t$

$\cos(\omega t \pm 180^\circ) = -\cos \omega t$

$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$

$\cos(\omega t \pm 90^\circ) = \sin \omega t$

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Note that the frequency of the phasor is not explicitly shown in the phasor diagram

For this reason phasor domain is also known as frequency domain.

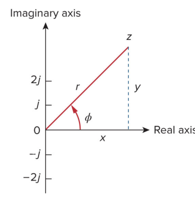
The derivative of *v(t)* [time domain] is transformed to the phasor domain as $j\omega$ **V**.

The integral of *v(t)* [time domain] is transformed to the phasor domain as **V** / $j\omega$.

A couple of other identities to remember/know:

$$-1 = 1\angle \pm 180^o$$
$$j = 1\angle 90^o$$

Consider what -j would be.



The differences between *v(t)* and **V** are important to remember:

- *v(t)* is the instantaneous or time domain representation while **V** is the frequency or phasor domain representation.
- *v(t)* is time dependent while **V** is not.
- *v(t)* is always real while **V** is generally complex.

Also note that phasor analysis applies only when:

- frequency is constant.
- manipulating two or more sinusoidal signals of the same frequency.

Examples and Practice Problems

E9.3 Evaluate the complex numbers: (a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $[(10\angle -30^\circ + (3 - j4)]/[(2 + j4)(3 - j5)^*]$

P9.3 Evaluate the complex numbers: (a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $[(10 + j5 + 3\angle 40^\circ)/(-3 + j4)] + 10\angle 30^\circ + j5$

E9.4 Transform these sinusoids to phasors: (a) $i = 6 \cos(50t - 40^\circ)$ A

(b) $v = -4 \sin(30t + 50^\circ)$ V

P9.4 Express these sinusoids as phasors: (a) $v = -14 \sin(5t - 22^\circ)$ V

(b) $i = -8 \cos(16t + 15^\circ)$ A

Examples and Practice Problems (Cont.)

E9.5 Find the sinusoids represented by these phasors: (a) $\mathbf{I} = -3 + j4$ A

(b) $\mathbf{V} = j8e^{-j20^\circ}$

P9.5 Find the sinusoids corresponding to these phasors: (a) $\mathbf{V} = -25\angle 40^\circ$

(b) $\mathbf{I} = j(12 - j5)$ A

E9.6 Given $i_1(t) = 4 \cos(\omega t + 30^\circ)$ A and $i_2(t) = 5 \sin(\omega t - 20^\circ)$ A, find their sum.

P9.6 Find $v = v_1 + v_2$, if $v_1 = -10 \sin(\omega t - 30^\circ)$ and $v_2 = 20 \cos(\omega t + 45^\circ)$

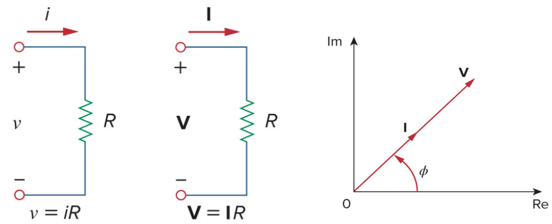
9.4 Phasor Relationships for Circuit Elements

Each circuit element has a relationship between its current and voltage.

These can be mapped into phasor relationships very simply for resistors, capacitors and inductors.

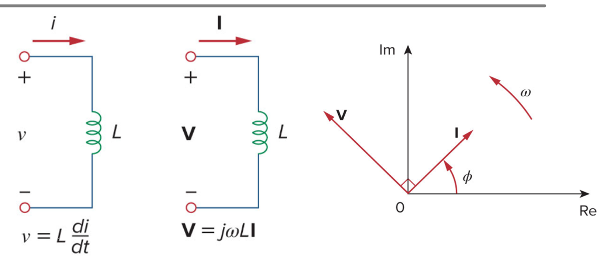
For the resistor, the voltage and current are related via Ohm's law.

As such, the voltage and current are in phase with each other.



Inductors have a phase shift of 90° between the voltage and current, and the standard convention is to say that the current lags the voltage.

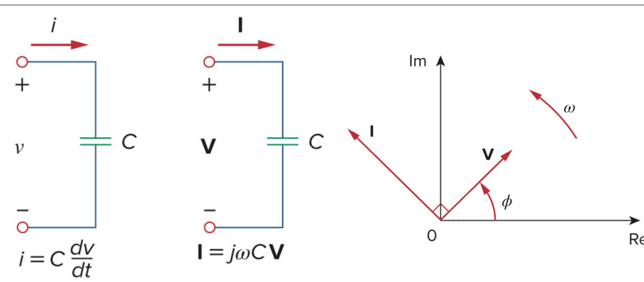
This is represented on the phasor diagram by a positive phase angle between the voltage and current.



Ch 9, p15

Capacitors have the opposite phase relationship as compared to inductors and the current leads the voltage.

In a phasor diagram, this corresponds to a negative phase angle between the voltage and current.



Here's a summary of the time-domain and phasor-domain representations of the current and voltage through/across the passive circuit elements.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

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Ch 9, p16

9.5 Impedance and Admittance

It is possible to expand Ohm's law ($v = iR$) to capacitors and inductors. Since the ratio of voltage and current across/through capacitors and inductors is always changing, doing this in the time domain would be difficult. However, doing this in the frequency domain is much easier.

Element	Frequency domain	$\dfrac{\mathbf{V}}{\mathbf{I}}$
R	$\mathbf{V} = R\mathbf{I}$	$= R$
L	$\mathbf{V} = j\omega L\mathbf{I}$	$= j\omega L$
C	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$	$= \dfrac{1}{j\omega C}$

The *impedance* (**Z**) of a circuit element is the ratio of the phasor voltage (**V**) to the phasor current (**I**):

Z= V/I or **V= Z I**

where **Z** is a frequency dependent quantity and is measured in ohms.

Admittance is simply the inverse of impedance and is measured in Siemens.

Here's a summary of the impedances and admittances of the passive circuit elements.

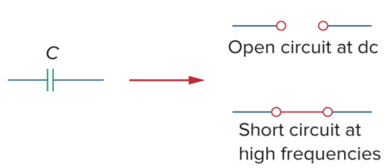
Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

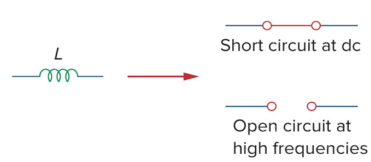
It is important to realize that in the frequency domain, the values obtained for impedance are only valid at that frequency.

Changing to a new frequency requires recalculating the values.

Look at the effect of angular frequency and, by extension, (cyclic) frequency on these relationships remembering that:

$$\omega T = 2\pi \qquad T = \frac{2\pi}{\omega} \qquad f = \frac{1}{T} = \frac{\omega}{2\pi}$$





As a complex quantity, impedance may be expressed in rectangular form, and the separation of the real and imaginary components is useful:

- the real part is the resistance.
- the imaginary component is called the ***reactance, X***. When it is positive, we say the impedance is inductive, and capacitive when it is negative.

Admittance, being the reciprocal of the impedance, is also a complex number.

- the real part of the admittance is called the *conductance*, *G*
- the imaginary part is called the *susceptance*, *B*

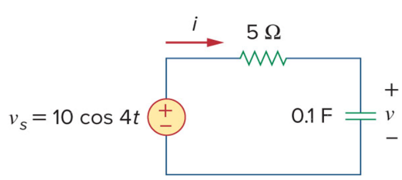
These are all expressed in Siemens or (mhos)

The impedance and admittance components can be related to each other:

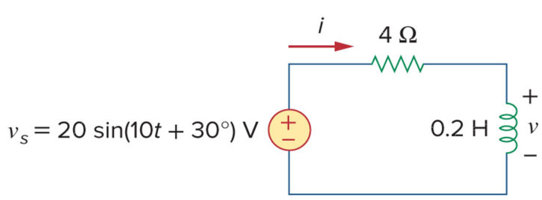
$$G = \frac{R}{R^2 + X^2} \qquad B = -\frac{X}{R^2 + X^2}$$

Examples and Practice Problems

E9.9 Find *v(t)* and *i(t)* in this circuit:



P9.9 Find *v(t)* and *i(t)* in this circuit:

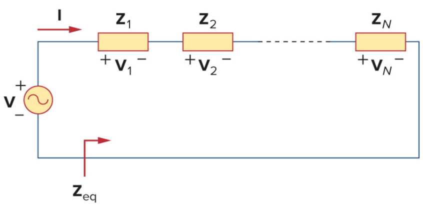


9.7 Impedance Combinations

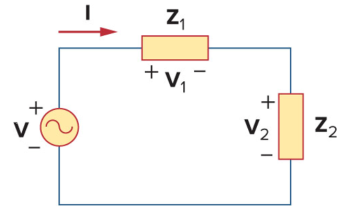
Impedance combinations follow the rules for resistors:

Series combinations will result in a sum of the impedance elements:

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \ldots + Z_N$$



Two elements in series can act like a voltage divider



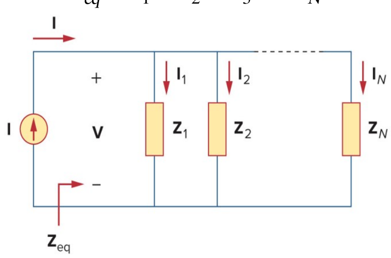
$$V_1 = \frac{Z_1}{Z_1+Z_2}V \qquad V_2 = \frac{Z_2}{Z_1+Z_2}V$$

NOTE: These relationships are expressed in terms of phasors

Ch 9, p21

Likewise, elements combined in parallel will combine in the same fashion as resistors in parallel:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \ldots \frac{1}{Z_N}$$



Expressed as admittance, though, they are again a sum:

$$Y_{eq} = Y_1 + Y_2 + Y_3 + \ldots + Y_N$$

Once again, these elements can act as a current divider:

$$I_1 = \frac{Z_2}{Z_1+Z_2}I \qquad I_2 = \frac{Z_1}{Z_1+Z_2}I$$

NOTE: These relationships are expressed in terms of phasors

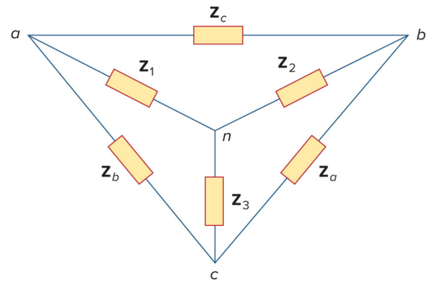
Ch 9, p22

The delta-to-wye and wye-to-delta transformations applicable to resistive circuits are also valid for impedances.

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c} \qquad Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c} \qquad Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} \qquad Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

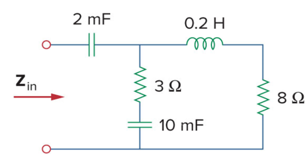


NOTE: These relationships are expressed in terms of phasors

Example and Practice Problems

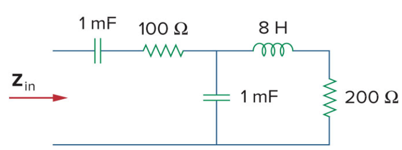
E9.10 Find the input impedance for the circuit shown:

(Assume the circuit operates at 50 rad/s.)

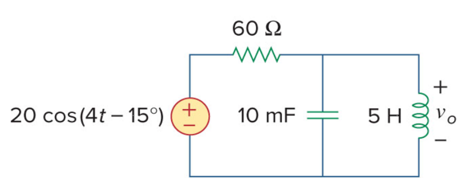


P9.10 Find the input impedance for the circuit shown:

(Assume the circuit is operating at 20 rad/s.)

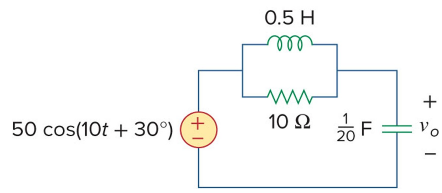


E9.11 Find $v_o(t)$ in this circuit.



Example and Practice Problems (Cont.)

P9.11 Calculate v_o in this circuit:



E9.12 Find current **I** in the circuit shown:

(Delta-Wye transformation)