Chapter 12

Local Weak Property Realism: Consistent Histories

Earlier, we surmised that weak property realism can escape the strictures of Bell's theorem and KS. In this chapter, we look at an interpretation of quantum mechanics that does just that, namely, the Consistent Histories Interpretation (CH).

12.1 Consistent Histories

The basic idea behind CH is that quantum mechanics is a stochastic theory operating on quantum histories. A history *a* is a temporal sequence of properties held by a system. For example, if we just consider spin, for an electron a history could be as follows: at t_0, $S_z = 1$; at t_1, $S_x = 1$; at t_2, $S_y = -1$, where spin components are measured in units of $\hbar/2$. The crucial point is that in this interpretation the electron is taken to exist independently of us and to *have* such spin components at such times, while in the standard view 1, 1, and -1 are just experimental return values. In other words, while the standard interpretation tries to provide the probability that such returns would be obtained upon measurement, CH provides the probability Pr(*a*) that the system will have history *a*, that is, such and such properties at different times. Hence, CH adopts non-contextual property realism (albeit of the weak sort) and dethrones measurement from the pivotal position it occupies in the orthodox interpretation. To see how CH works, we need first to look at some of the formal features of probability.

12.2 Probability

Probability is defined on a *sample space* S, the set of all the mutually exclusive outcomes of a state of affairs; each element *e* of S is a *sample point* to which one assigns a number Pr(*e*) satisfying the axioms of probability (of which more later). An *event* is a set of one or more sample points (a subset of S). If S contains *n* elements, then the set of all events, the *event algebra*, has 2^n elements, including the empty set $\varnothing$ (the set with no points) and S itself. In other words, the event algebra is the set of all the subsets of S. Given an event *A*, $\sim A$ is the *complement* of *A*, namely, the event of all the points in *S* that do not belong to *A*, that is, the event that occurs if and only if *A* does not occur.

EXAMPLE 12.2.1

Consider the tossing of a fair die. Then, $S = \{1,2,3,4,5,6\}$; $\{1\}$ is the event '1 appears on top' and it has probability 1/6; $\{3,5\}$ is the event '3 and 5 (in whatever order) appear on top on two successive tosses' and it has probability 1/18; *S* is, for example, the event 'a number appears on top' and $\{\varnothing\}$ is, for example, the event 'no number appears on top.' Finally, if $A = \{3,5\}$, then $\sim A = \{1,2,4,6\}$.

Given the sample space *S* and two events *A* and *B* on *S*, one can define two operations, disjunction and conjunction, as follows.

$$C = A \cup B, \tag{12.2.1}$$

the *disjunction* of *A* and *B*, is the event of all the points that belong to *A* and/or *B*. For example, if $A = \{3,5\}$ and $B = \{1,5\}$, then $C = \{1,3,5\}$. *C* occurs if and only if *A* and/or *B* occurs.

$$D = A \cap B, \tag{12.2.2}$$

the *conjunction* of *A* and *B*, is the event of all the points belonging to both *A* and *B*. *D* occurs if and only if both *A* and *B* occur. Disjunction and conjunction satisfy the following laws L1-L4:

L1, or Commutative Law: $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

L2, or Associative Law: $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$.

L3, or Distributive Law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$ and

$A \cap (B \cup C) = (A \cap B) \cup (A \cap B)$.

L4, or Identity Law: there exist two events $\{\varnothing\}$ and S such that $A \cup \{\varnothing\} = A$ and

$A \cap S = A$.

Any structure on which one can define disjunction, conjunction, and complement in such a way as to satisfy L1-L4 is a *Boolean algebra*. Many common algebras such as set theory and propositional logic, are Boolean algebras. What we are going to do now is to construct a Boolean algebra in Hilbert space.

EXAMPLE 12.2.2

Suppose we toss two fair dice simultaneously. Then, $3 \cup 5$, (3 or 5 come up), obtains just in case any, or all, of those numbers comes up.[1] By contrast, $1 \cap 4$ (1 and 4 come up) is true just in case both 1 and 4 come up. Note that $3 \cup 4$ says exactly the same thing as (is equivalent to) $\sim(\sim 3 \cap \sim 4)$, that is, it is false that neither 3 nor 4 come up.

12.3 Properties at a Time

Given a Hilbert space H, assumed to be *finite*, a projector P identifies a linear subspace Π of H composed of all and only the eigenkets $|\chi\rangle$ of P such that $P|\chi\rangle = |\chi\rangle$.[2] Hence, the generic projector $P = |\psi\rangle\langle\psi|$ operates on the eigenkets in Π as the identity operator: $P|\psi\rangle = |\psi\rangle\langle\psi|\psi\rangle = |\psi\rangle$. When $\Pi = H$, then P is the identity operator I that transforms every ket in H into intself. If $\{\psi_1, ..., \psi_n\}$ is an orthonormal basis of H, then

$$I = \sum_i |\psi_i\rangle\langle\psi_i|. \qquad (12.3.1)$$

[1] Since $3 \cup 5$ obtains if and only if '$3 \cup 5$' is true, we use the two interchangeably.

[2] Here, as in the remainder of this exposition of CH, we follow Griffiths, R., (2002).

Clearly, if $|\phi\rangle$ is orthogonal to the members of Π, then it is an eigenket of *P* but it does not belong to Π because $P|\phi\rangle = |\psi\rangle\langle\psi|\phi\rangle = 0$. Given a projector *P*,

$$\sim P = I - P \tag{12.3.2}$$

is *P*'s complement.

A physical property is something that can be predicated of a physical system at a time. For example, 'the *z*-component of spin is 1' is a physical property. Note that a physical property is always associated with a value. So, S_z is *not* a physical property, but $S_z = 1$ is. If a system S is described by a ket $|\chi\rangle$ in Π (that is, $P|\chi\rangle = |\chi\rangle$), then one can say that S has the property P standardly associated with $|\chi\rangle$.[3] If $|\chi\rangle$ is orthogonal to the kets in Π (that is, $P|\chi\rangle = 0$), then S has ~P (the negation of the property P), namely, the property associated with the projector ~P. If $|\chi\rangle$ is not an eigenket of *P*, then the property P is undefined. It follows that a property associated with *I* always holds and one associated with the subspace made up of the zero vector never holds. Two projectors *P* and *Q* are orthogonal if

[3] Here we denote both the property and its projector as *P*. Of course, they are not the same thing: properties exist in the real world, while projectors are abstract entities existing in configuration space. However, since for simplicity we assume that there is no degeneracy, there is a one-one correspondence between properties and projectors. At times, however, we shall denote with $[P]$ the projector associated with property *P*.

$$PQ = QP = 0.$$[4] (12.3.3)

A decomposition of the identity operator *I* is a collection of orthogonal projectors R_i such that

$$I = \sum_i R_i. \tag{12.3.4}$$

We can logically link properties by using the following two connectives. The first connective is conjunction, symbolized by '$\cap$', which stands for 'and', so that '$P \cap Q$' means P and Q. Rule Q1 says that quantum-mechanically this is represented by $[P]\cdot[Q]$, where $[P]$ is the projector associated with P and $[Q]$ that associated with Q.

The second connective is disjunction, symbolized by '$\cup$', which stands for 'and/or', so that '$P \cup Q$' means P and/or Q. Rule Q2 says that quantum-mechanically disjunction is represented by $[P]+[Q]-[P]\cdot[Q]$. Crucially, $P \cap Q$ and $P \cup Q$ are defined *only if* $[P]$ and $[Q]$ commute. It can be shown that a decomposition of the identity projector *I* constitutes a sample space and that the event algebra is a Boolean algebra under the operations of conjunction and disjunction. All of this looks confusing, but, as the following example will show, there is really less than meets the eye.

EXAMPLE 12.3.1

Consider a spin-half particle in space H. The identity projector on H is $I = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$, the sum of orthogonal, and therefore commuting, projectors. Consider now the projector $P = |\uparrow\rangle\langle\uparrow|$. Then, $\Pi = |\uparrow\rangle$ and the property P is $S_z = 1$. If the particle is

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[4] One must not confuse the orthogonality between vectors (their inner product is zero) and between projectors (their product is zero). Obviously, orthogonal projectors commute.

in state $|\uparrow\rangle$, then we can say that it has the property $S_z = 1$. If the particle is in state $|\downarrow\rangle$, which is orthogonal to $|\uparrow\rangle$, then it has the negation of $S_z = 1$, namely, $S_z = -1$. The reason is that $I - P = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| - |\uparrow\rangle\langle\uparrow| = |\downarrow\rangle\langle\downarrow| =\sim P$ and $S_z = -1$ is associated with $|\downarrow\rangle$. If the particle is in state $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$, then $S_z = 1$ is undefined. The property associated with I is '$S_z = 1$ or $S_z = -1$' and it always hold, while the property '$S_z = 1$ and $S_z = -1$' never does. The properties '$(S_z = 1) \cap (S_x = 1)$' and '$(S_z = 1) \cup (S_x = 1)$' are undefined.[5] Now let us suppose that particle a is in spin-state $|\uparrow\rangle$, that it moves, and that its position at x corresponds to the state $|x\rangle$. Using obvious notation, Q1 says that the quantum mechanical representation of '$S_z = 1 \cap x = 1$' is $(|\uparrow\rangle\langle\uparrow|)(|1\rangle\langle 1|)$, since the two operators commute. Moreover, Q2 says that the quantum mechanical representations for '$S_z = 1 \cup x = 1$' is $|\uparrow\rangle\langle\uparrow| + |1\rangle\langle 1| - (|\uparrow\rangle\langle\uparrow|)(|1\rangle\langle 1|)$.

12.4 Non-existent Properties

Although CH allows a realist understanding of quantum mechanics, it does not follow EPR in attributing quantum mechanically incompatible properties to a system. Griffiths gives an instructive story about what happens if one insists that $P \cap Q$ is defined even if P and Q are incompatible. Consider a spin-half particle, and to simplify the notation, let $[Z+]$ stand for the projector associated with $S_z = 1$, and similarly for other projectors. Suppose now that the composite property $S_z = 1 \cap S_x = 1$ existed. Then its corresponding projector would have to project onto a subspace Π of the two-dimensional Hilbert space H of the spin-half particle. However, no such subspace can exist. For, no one-dimensional subspace $|\uparrow_i\rangle$ is associated with both $S_z = 1$ and $S_x = 1$; all

[5] In order to avoid clutter, in the future we shall omit the brackets when possible.

the one-dimensional subspaces are, as it were, already taken. This leaves only two subspaces, namely, H itself and the zero-dimensional subspace 0 containing only the zero vector. Since neither $S_z = 1$ nor $S_x = 1$ are always true, $\Pi \neq H$. If $\Pi = 0$, then

$$S_z = 1 \cap S_x = 1 \tag{12.4.1}$$

could never be true, and consequently

$$S_z = 1 \cap S_x = -1 \tag{12.4.2}$$

could never be true as well. Since the disjunction of two false sentences is false,

$$(S_z = 1 \cap S_x = 1) \cup (S_z = 1 \cap S_x = -1) \tag{12.4.3}$$

is false as well. Now (12.4.3) is logically equivalent to

$$S_z = 1 \cap (S_x = 1 \cup S_x = -1) \tag{12.4.4}$$

which must, therefore, be always false. However, the part in brackets is always true because the corresponding projector is the identity projector *I* in H, and consequently $S_z = 1$ must always be false. But this cannot be right because at times $S_z = 1$ is true. If we assume that the Hilbert space H contains all the information about the particle, that is, *if* we assume that quantum mechanics is complete, then $S_z = 1 \cap S_x = 1$ cannot be assigned any meaning at all because nothing in H can be associated with it. Moreover, since conjunction can be defined in terms of disjunction plus negation and vice versa, $S_z = 1 \cup S_x = 1$ is meaningless as well. In sum, the penalty for logically connecting non-commuting properties is loss of meaning.

The commuting requirement has interesting consequences for the measurement problem. As we saw earlier, the measurement problem consists in the fact that according to TDSE the outcome of measurement is a superposition like

$$|\Psi'\rangle = \sum_i c_i |\psi_i\rangle \otimes |\chi_i\rangle, \qquad (12.4.5)$$

where $|\psi_i\rangle$ and $|\chi_i\rangle$ are eigenstates of the observed system and the measuring device respectively. However, we should note that $|\Psi'\rangle\langle\Psi'|$ does not commute with any $|\chi_i\rangle\langle\chi_i|$, and consequently once $|\Psi'\rangle\langle\Psi'|$ obtains it is meaningless to ask whether any of the $|\chi_i\rangle\langle\chi_i|$, or any disjunction of the $|\chi_i\rangle\langle\chi_i|$, obtain. In other words, once the combination atom-Geiger counter-hammer-cyanide container-cat has reached the measurement superposition TDSE entails, it makes no sense to ask whether the cat is alive or dead, or even to say that the cat is alive or dead.

12.5 Quantum Histories

Consider a system S and its configuration space H. A history *a* is a tensor product of projectors of H such that

$$a = [\mathrm{P}_1] \otimes ... \otimes [P_n], \qquad (12.5.1)$$

where $[P_i]$ is the projector associated with the property P_i. History *a* is itself a projector in the history Hilbert space $\breve{\mathrm{H}} = \mathrm{H}_1 \otimes ... \otimes \mathrm{H}_n$, where H_j is S's configuration space at time t_j. Intuitively, *a* says that S holds properties $\mathrm{P}_1,\ldots,\mathrm{P}_n$ at times $t_1,...,t_n$. As a sample space for properties at one time is a decomposition of *I*, the identity projector for H, into mutually orthogonal property projectors, so a sample space for histories is a decomposition of $\breve{I}$, the identity projector for $\breve{\mathrm{H}}$, into mutually orthogonal history projectors, so that

$$\sum_a a = \breve{I}. \qquad (12.5.2)$$

The rules for negation, conjunction, and disjunction for histories are the same as those for properties at a single time. Hence, given two histories *a* and *b*,

$$\begin{aligned} &\sim a = \breve{I} - a \\ &a \cap b = ab \qquad (12.5.3) \\ &a \cup b = a + b - ab. \end{aligned}$$

As before, conjunction and disjunction are defined only if *a* and *b* commute, that is, only if $ab = ba$. Typically, *a* and *b* commute only if all of their component projectors associated with the same time commute; however, if the two histories are orthogonal ($ab = ba = 0$), then they commute even if not all of their component projectors associated with the same time commute. The histories that are sample points are *elementary* histories, while their combinations in terms of conjunction, disjunction, and negation (events of more than one sample point) are compound histories. Since they are orthogonal, elementary histories are mutually exclusive, and therefore they differ from each other by at least one property projector. The event algebra is a Boolean algebra under the operations of conjunction and disjunction.

EXAMPLE 12.4.1

Let $a = [\mathrm{X}+] \otimes [\mathrm{Z}+] \otimes [\mathrm{Y}-]$ and $b = [\mathrm{Y}+] \otimes [\mathrm{Z}+] \otimes [X-]$ be two histories of the spin-half system S made of one particle.[6] Then $a \cap b$ and $a \cup b$ are not defined because *a* and *b* do not commute, since the first and third projectors do not commute and the two histories are not orthogonal.

The *chain operator* for a history *a* is

$$C_a = P_n U(t_n, t_{n-1}) P_{n-1} \cdots U(t_1, t_0) P_0, \qquad (12.5.4)$$

where $P_0, \ldots, P_n$ are the property projectors of *a* and $U(t_i, t_j)$ is the time evolution

[6] To avoid clutter, we often dispense with the subscripts when the temporal ordering of the system's properties is clear.

operator.[7] The probability of a history occurring is given by

$$\Pr(a) = Tr\left(C_a C^+{}_a\right), \tag{12.5.5}$$

where $C^+{}_a$ is the adjoint of C_a.[8]

EXAMPLE 12.4.2

Let us determine the probability that an electron originally in state $\left|\uparrow_z\right\rangle$ will have the history $a = [\mathrm{Z}\ +] \otimes [\mathrm{X}\ +] \otimes [\mathrm{Y}\ -]$ if the Hamiltonian is zero. A simple calculation gives $[\mathrm{Y}\ -] = \left|\downarrow_y\right\rangle\left\langle\downarrow_y\right| = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$; similarly, $[\mathrm{X}\ +] = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $[\mathrm{Z}\ +] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Consequently, the chain history operator is

$$C_a = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+i & 0 \\ 1-i & 0 \end{pmatrix}.$$

Finally,

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[7] Notice that the projector corresponding to the earliest time is on the right. Note also that when the Hamiltonian is zero, the evolution operator becomes the identity operator, and can therefore be ignored.

[8] Actually, $Tr\left(C_a C^+{}_a\right)$ is not the probability, but the weight (the unnormalized probability, as it were) of *a*. The general formula is $\Pr(a) = \dfrac{Tr\left(C_a C^+{}_a\right)}{Tr[P_0]}$, where $[P_0]$ is the initial projector of the history. However, since we are dealing only with orthonormal bases and pure states, $[P_0]$ always projects onto a one-dimensional subspace. Hence, its trace is always equal to one, and therefore we can use the simpler formula. In the literature, there are several equivalent formulas for history probability. We look at some of them in exercise 12.5.

$$\Pr(a) = Tr\left[\frac{1}{4}\begin{pmatrix} 1+i & 0 \\ 1-i & 0 \end{pmatrix}\frac{1}{4}\begin{pmatrix} 1-i & 1+i \\ 0 & 0 \end{pmatrix}\right] = \frac{1}{4}.$$

The idea now is to think of quantum mechanics as a stochastic or probabilistic theory: (12.5.5), which is equivalent to Wigner's formula, is used to assign probabilities to quantum histories much in the same way in which classical stochastic theories assign probabilities to sequences of coin tosses or even to a truly indeterministic sequence of events.[9]

12.6 Restrictions on Histories

At this point, however, we must make sure that what we introduced in (12.5.5) is really a probability, that is, it satisfies the appropriate axioms. There are many equivalent axiomatic formulation of probability. Here is a simple one consisting of three axioms:

1. Positivity: $0 \leq \Pr(a)$;
2. Additivity: if a and b are any two mutually exclusive events, then $\Pr(a \vee b) = \Pr(a) + \Pr(b)$;
3. Normalization: $\sum_e \Pr(e) = 1$, where e is a sample point.

We can show that (12.5.5) satisfies positivity. An operator O is positive just in case the elements of its main diagonal are positive. However, given any O such that $O|\psi\rangle = |\varphi\rangle$,

$$\langle\psi|\tilde{O}^* O|\psi\rangle = \langle\varphi|\varphi\rangle \geq 0, \qquad (12.6.1)$$

and therefore $\tilde{O}^* O$ is a positive operator, which entails that its trace is positive.

However, it turns out that (12.5.5) does not always satisfy additivity because of interference effects among histories, and this requires that the Boolean algebras of events on which (12.5.5) can be applied must be restricted: histories cannot be lumped together

[9] For Wigner's formula, see appendix five.

haphazardly if we want them to obey the rules of probability. The restriction amounts to the elimination of interference effects so that any two mutually exclusive histories can evolve separately. As a result, probabilities are assigned only to histories belonging to a *family* F if

$$Tr\left(C^{+}{}_{a}C_{b}\right)=0, \qquad (12.6.2)$$

where *a* and *b* are *any* two different histories belonging to F. In effect, (12.6.2) guarantees additivity for history probabilities. If, following Griffiths, we define the *inner product* for operators as

$$\langle A,B\rangle = Tr\left(A^{+}B\right), \qquad (12.6.3)$$

then *a family is consistent if its history chain operators are orthogonal*.[10]

It can also be shown that (12.5.5) satisfies normalization.

EXAMPLE 12.6.1

Let us investigate whether the three spin-half histories with zero Hamiltonian

$$a=[\mathrm{Z}+]\otimes[\mathrm{Y}+]\otimes[\mathrm{X}+],$$

$$b=[\mathrm{Z}-]\otimes[\mathrm{Y}+]\otimes[\mathrm{X}-],$$

$$c=[\mathrm{Z}+]\otimes[\mathrm{Y}-]\otimes[\mathrm{X}+]$$

form a consistent family. Since $Tr\left(\tilde{C}_{a}^{*}C_{c}\right)\neq 0$, and *a*, *b*, and *c* fail to form a consistent family. Note that the orthogonality of *a* and *c* does not entail that their chain operators are orthogonal as well.

Determining family consistency can be very laborious, especially when it comes

[10] For a proof that (12.6.2) entails additivity, see Griffiths, R., (2002): 140. It turns out that (12.6.2) is a sufficient but not a necessary condition for additivity.

to checking the orthogonality of history chain operators. Nevertheless, there are shortcuts. One is that if two histories have *orthogonal first or last* property projectors, then their chain operators are orthogonal as well. Hence, the spin-half family made up of

$$d = [\mathrm{Z}+] \otimes [\mathrm{Y}+] \otimes [\mathrm{X}+],$$

$$e = [\mathrm{Z}-] \otimes [\mathrm{Y}+] \otimes [\mathrm{X}+], \qquad (12.6.4)$$

$$f = [\mathrm{Z}+] \otimes [\mathrm{Y}-] \otimes [\mathrm{X}-]$$

is consistent because any two of the histories have orthogonal first or last members. For example, the first members of *d* and *e* are orthogonal since $[\mathrm{Z}+] \cdot [\mathrm{Z}-] = 0$.

EXAMPLE 12.6.2

We can use the above shortcut to come up with a family of histories for a simple measurement. Suppose we shoot a spin-half particle in state $|\uparrow_x\rangle$ through a SGZ device and that the particle goes through positions λ_1 and λ_2 on its way to the device, eventually emerging from it in position $\lambda+$ or $\lambda-$. If $\lambda+$ and $\lambda-$ are sufficiently far, so that the wave packets overlap only minimally, then the respective vector states will be effectively orthogonal, thus representing mutually exclusive alternatives. Hence, we may consider $\lambda+$ and $\lambda-$ as 'pointers' correlated with the values of S_z. One can then construct the following family made up of two histories

$$a = [X+][\lambda_1] \otimes [Z+][\lambda_2] \otimes [Z+][\lambda+]$$

and

$$b = [X+][\lambda_1] \otimes [Z-][\lambda_2] \otimes [Z-][\lambda-].$$ [11]

[11] Actually, the family is not complete, and one should add $c = \{I - [X+][\lambda_1]\} \otimes I \otimes I$. However, since we may set $\Pr(c) = 0$, this history can be discarded without any loss.

The two stories are orthogonal because their last members are, and they show the measurement correlations between z-spin values and particle position.

12.7 The Risks of Joining Families and Boolean Algebras

Typically, families cannot be mixed: jumping from a family to another in the same description of a physical process is forbidden. This prohibition marks most clearly the difference between the quantum and the classical world, and therefore we should look at it a bit more closely. Two elementary histories in the same family are incompatible in the sense of being mutually exclusive: if one is true, the others must be false. In other words, the penalty for combining them is logical inconsistency, a statement of the form $A \cap \sim A$. This is not peculiar to quantum mechanics; if it is true that that at a given time the x-component of the spin of an electron or a billiard ball has a certain value, than it is false that at that same time it has a different value.

However, histories belonging to different families are incompatible in the sense that their conjunction generates not a logically inconsistent statement but one that is neither true nor false, that is, a string of symbols that is no statement at all. To put it differently, the penalty for combining histories from different families is meaninglessness. The reason is that every consistent family constitutes a Boolean algebra of history projectors that acquire probabilities only within *that* algebraic framework since, properly speaking, any sort of probabilistic reasoning depends on a sample space. Quantum mechanics is peculiar in that different algebras of histories cannot be joined unless certain conditions, ultimately arising from the fact that operator multiplication is not commutative, are satisfied. So, absent such conditions, if a and b belong to different families, any statement involving both of them has no probability attached to it (no probability can be defined for it) and is therefore meaningless. In sum, quantum theoretical statements are probabilistic ("true" means 'having probability one' and "false" means 'having probability zero'), and therefore they presuppose a sample space. However, typically the non-commutative property of operator multiplication prevents the construction of a single sample space from those of two families unless certain special conditions are satisfied. It can also be shown that the prohibition against joining families allows the adoption of non-contextual weak value determinism without impinging on the KS theorem (Griffiths, R., (2002): ch. 22).

12.8 A Comparison of CH with the Orthodox Interpretation

Griffiths and Omnès use a thought experiment with a Mach-Zehnder interferometer to highlight the advantages of CH over the standard interpretation.[12] A photon initially at *a* goes through a beam splitter S_1, is deflected by the mirrors M_1 and M_2, and eventually is detected by C or D, which undergo a macroscopic change of state by, say, clicking. The second beam splitter S_2 is not present in this run.

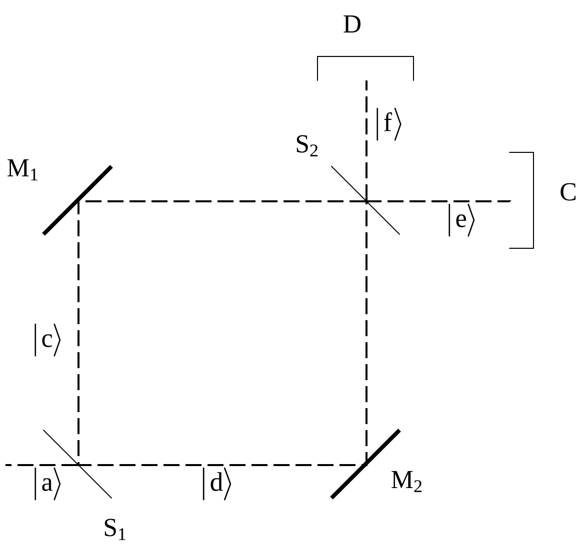

Figure 1

[12] Griffiths, R., and Omnès, R., (1999).

Using obvious symbolism, the orthodox representation of this story is as follows:

$$|a\rangle \otimes |C\rangle \otimes |D\rangle \Rightarrow \frac{1}{\sqrt{2}}\left(|c\rangle + |d\rangle\right) \otimes |C\rangle \otimes |D\rangle \Rightarrow \frac{1}{\sqrt{2}}\left(\left|C^*\right\rangle \otimes |D\rangle + |C\rangle \otimes \left|D^*\right\rangle\right) \quad (12.8.1)$$

where $\left|C^*\right\rangle$ is the state of C clicking, and the arrows indicate the linear temporal development provided by TDSE. As soon as the system arrives at entanglement between particle and detector, the state function collapses and one and only one of the two alternatives is realized. Supposing that the final state of the system is, for example, $\left|C^*\right\rangle \otimes |D\rangle$, one still cannot know what happened before because of the bizarre superposition in the intermediate step. In other words, retrodiction from experimental results is impossible. However, Griffiths and Omnès note, retrodiction is constantly used by particle physicists, who assume not only that measurements accurately reflect the state of affairs just before measurement, but even extrapolate which path a particle has followed before the measurement.

By contrast, CH could employ the two mutually exclusive histories

$$aCD \otimes cCD \otimes C^*D \quad (12.8.2)$$

and

$$aCD \otimes dCD \otimes CD^*, \quad (12.8.3)$$

to each of which it attributes probability 1/2.[13] Here, not only is the collapse postulate unnecessary, but if the final state of the system is C^*D, we know which history has been actualized, and therefore we can say that the particle went along path *c*, with the result that we need not say, as Griffiths and Omnès flippantly put it, that "experimenters don't take enough courses in quantum theory" (Griffiths, R., and Omnès, R., (1999): 29).

[13] As before, to avoid clutter, we occasionally write P instead of $[P]$.

It is true that CH also allows histories in which it is meaningless to ask which path the photon followed, such as

$$aCD \otimes \frac{1}{\sqrt{2}}[c+d]CD \otimes C^{*}D \tag{12.8.4}$$

and its counterpart in which D clicks. However, one need not use them. Hence, while in classical physics a single description allows one to answer all the meaningful questions about a system, largely in quantum mechanics what counts as a meaningful question depends on the description employed.

Suppose now that we insert a second beam splitter S_2 at the intersection of paths *c* and *d* near the detectors *C* and *D*, and that we alter the optical path lengths so that S_2 will produce the unitary transitions

$$|c\rangle \Rightarrow \frac{1}{\sqrt{2}}\left(|e\rangle + |f\rangle\right) \tag{12.8.5}$$

and

$$|d\rangle \Rightarrow \frac{1}{\sqrt{2}}\left(-|e\rangle + |f\rangle\right), \tag{12.8.6}$$

with the result that the two $|e\rangle$'s have opposite phases, and therefore cancel each other out. The outcome is that all the photons will always travel along *f* and never along *e*, so that *D* will record hits all the times. At this point, our classical intuitions push us to wonder whether before getting along *f* the photon went along *c* or *d*. The corresponding histories are

$$h = [a] \otimes [c] \otimes [f] \tag{12.8.7}$$

and

$$i = [a] \otimes [d] \otimes [f]. \tag{12.8.8}$$

Can the two histories be logically connected? For example, can one sensibly ask whether $(h \cup i) \cap \sim (h \cap i)$, that is, the photon went along *c* or *d* but not both, or even sensibly state that $h \cap i$, that is, the photon went along both *c* and *d*, as one occasionally reads? According to CH, the answer is negative because the chain operators for the two histories are not orthogonal, and therefore h and *i* cannot belong in the same consistent family.

12.9 The Histories of EPR

In order to discuss the EPR paper with the help of CH, we need to introduce two notions, that of history extension, and that of support of a family. Moreover, we need to find out what are the conditions for joining two families into a new family. Suppose that $a = P_1 \otimes ... \otimes P_n$ and we want to extend *a* to a time later than t_n. All we need to do is to set *I*, the identity operator, for that time: $a = P_1 \otimes ... \otimes P_n \otimes I_{n+1}$. Since *I* corresponds to a property that is always true, in effect we have added nothing to our original story, although, formally, it now extends to a time it did not cover before. The same procedure applies if the added time is anywhere in the history.

Consider now the histories

$$a = [\mathrm{Z}+] \otimes [\mathrm{Y}+] \otimes [\mathrm{X}+], \tag{12.9.1}$$

and

$$b = [\mathrm{Z}+] \otimes [\mathrm{Y}-] \otimes [\mathrm{X}-]. \tag{12.9.2}$$

Since $a + b \neq \breve{I}$, they do not constitute a complete family *F* and that another history *c* would have to be added to obtain *F*. However, it may turn out that they are the only histories we care about, in which we may set $\Pr(c) = 0$ and take into account only the *support* of *F*, namely all and only the histories whose probability is greater than zero.

Sometimes, families can be joined. Not surprisingly, two consistent families

$A = \{a_i\}$ and $B = \{b_j\}$ are compatible (can be joined) if and only if the following two conditions are satisfied.

First, the histories belonging to the two families must commute: for all *i* and *j*, $ab = ba$. This in effect guarantees that *a* and *b* can be linked by logical connectives like $a \cap b$ and $a \cup b$. Second, for all *a, b, c, d,* $\langle C_{ab}, C_{cd} \rangle = 0$, where $a \neq c$ belong to *A* and $b \neq d$ belong to *B*. In other words, the chain operators of the products of different histories from the two families must be orthogonal. This condition entails additivity. If $ab = 0$ or $cd = 0$ (if the histories in the two families are orthogonal), then the second condition is automatically satisfied.

We can now address EPR by utilizing histories as close as possible to what its proponents presumably had in mind. Consider, then the family *Z* with support

$$\Psi_0 Z_a^0 \otimes \begin{cases} z_a^+ Z_a^0 z_b^- & \otimes \quad Z_a^+ z_b^- \\ z_a^- Z_a^0 z_b^+ & \otimes \quad Z_a^- z_b^+ \end{cases}, \tag{12.9.3}$$

where $|Z_a^0\rangle$ is the initial state of an SGZ, z_a^+ indicates that particle *a* has the property $S_z = 1$, Z_a^+ indicates that the SGZ has recorded $S_z^a = 1$ by absorbing *a*, and similarly for the other symbols. Here we have two histories with a common beginning and a split denoted by the bracket. Note that particles *a* and *b* really have the appropriate *z*-spin value even before the measurement, and that the measurement correctly correlates the measurement returns on *a* with the unmeasured value of *z*-spin on *b*, and vice versa. Consequently, there is no need to appeal to non-locality.

Consider now the family *X* with support

$$\Psi_0 X_a^0 \otimes \begin{cases} x_a^+ X_a^0 x_b^- & \otimes \quad X_a^+ x_b^- \\ x_a^- X_a^0 x_b^+ & \otimes \quad X_a^- x_b^+ \end{cases}, \tag{12.9.4}$$

where the symbols have obvious meaning. To be allowed to say that *a* and *b* have both sharp *x*-spin and *z*-spin values, it must be possible for the two families *Z* and *X* to be joined to form a single family for, as we saw, connecting histories from different families may lead to meaniglessness. In this case, the two families can be joined because the histories in *Z* are orthogonal to those in *X*, since

$$\left(\Psi_0 X_a^0\right)\cdot\left(\Psi_0 Z_a^0\right)=0.^{14} \tag{12.9.5}$$

However, the joint probability of any history in *Z* with any history in *X* is zero, and therefore EPR was wrong in arguing that *a* and *b* have both sharp and opposite *x*-spin and *z*-spin values.

Still, CH seems to intimate that the probability that *a* and *b* have both sharp and opposite *x*-spin and *z*-spin values is zero. If so, one might suspect that Einstein lost the battle but won the war; since only meaningful sentences can have probability zero, one could infer that Bohr's position that talk of a particle having simultaneous non-commuting properties is meaningless is wrong. In other words, does the fact that *Z* and *X* can be joined into a single family ultimately vindicate Einstein's position that talk of a particle having simultaneous non-commuting properties makes perfect sense? If so, then the EPR paper was right in holding that quantum mechanics is incomplete since, as we saw above, there is no room in Hilbert space for the conjunction of non-commuting operators.

CH addresses the problem by introducing the notion of *dependent event*.

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[14] $\left[X_a^0\right]$ and $\left[Z_a^0\right]$ are orthogonal because a SGX and a SGZ differ macroscopically (their orientations are perpendicular). Since they are multiplied by the same operator, (12.9.5) holds.

Consider the family Z in its entirety. It consists of

$$\Psi_0 Z_a^0 \otimes \begin{cases} z_a^+ Z_a^0 z_b^- & \otimes \quad Z_a^+ z_b^- \\ z_a^- Z_a^0 z_b^+ & \otimes \quad Z_a^- z_b^+ \end{cases}, \tag{12.9.6}$$

namely, the support for Z, and a third history to which we assign probability zero, namely,

$$\left(I - \Psi_0 Z_a^0\right) \otimes I \otimes I. \tag{12.9.7}$$

Note that (12.9.7) says nothing at all about the spin components of a and b (I corresponds to the universal property). Hence Z's Boolean algebra requires that events $Z_a^+ z_b^-$ and $Z_a^- z_b^+$ be logically dependent on $\Psi_0 Z_a^0$ (the first event of the only histories in which they appear) in the sense that one may sensibly ask what is the probability of them obtaining only in the context of histories beginning with $\Psi_0 Z_a^0$. In other words, statements "$Z_a^+ z_b^-$" and "$Z_a^- z_b^+$" are meaningful only contextually, given that $\Psi_0 Z_a^0$ obtains. Similarly, $X_a^+ x_b^-$ and $X_a^- x_b^+$ depend on $\Psi_0 X_a^0$. However, since $\Psi_0 Z_a^0$ and $\Psi_0 X_a^0$ are mutually exclusive, it follows that "*b*'s *z*-spin component has such and such value" and "*b*'s *x*-spin component has such and such value" can be meaningful only in mutually exclusive contexts.[15]

12.10 A Review of CH

The basic idea behind CH is to consider quantum mechanics a classic stochastic theory providing the values of the physical properties a system actually has at a given time or at different times. Compliance with the laws of probability is obtained by the construction of Boolean algebras culminating in the notion of consistent family. CH lets

[15] Of course, in a way this is nothing new: the end of the EPR paper expressly argues against this move. CH has an interesting treatment of the counterfactuals apparently involved in the EPR paper; see section 24.2 of Griffiths, R., (2002).

one adopt non-contextual weak value determinism while avoiding non-locality, a feat that Bell's theorem had rendered doubtful, and it treats collapse as a mere computational device with no physical counterpart. Hence, measurement is dethroned from the central position it enjoys in the orthodox interpretation, and therefore all the (for many) unpalatable appeals to the observer or even to consciousness can be eliminated.

CH allows great flexibility in the description of a system because within the confines dictated by the consistency requirements the choice of the times and the properties with which a history deals is arbitrary. This has startling results not only because not all synchronous quantum histories of a system can be consistently combined (a feature present in classical histories as well), but also because synchronous quantum histories of a system belonging to incompatible families cannot be meaningfully combined. This entails that, in contrast to a classical system, a quantum one is not amenable to a single comprehensive description and therefore can be considered from different and typically non-combinable perspectives. However, the same physical question will receive the same answer in each different perspective, thus showing that none is more basic or closer to reality than any other.

It may prove helpful to think about CH in terms of the debate between Einstein and Bohr. CH partially agrees with Einstein in adopting property realism, albeit in a weakened form, and rejecting non-locality. However, contrary to the EPR paper, CH limits our ability to talk about a system by restricting it to consistent families, and with Bohr treats the attribution of non-commuting properties to a system as meaningless. In a way, CH can be seen as close to Bohr's views, for one might think of the idea of consistent families and their relations as a refinement of Bohr's idea of complementarity.

Nevertheless, in agreement with Einstein, CH dethrones measurement and the role of the observer from the central position they enjoy in standard quantum mechanics. True, what family one chooses in the description of a system is up to the physicist, but this is not different from the fact that in a photograph the point of view is up to the photographer. What is peculiar is that, because operator multiplication is not commutative, not all such photographs can be combined to produce a unique overall visual representation of the object.

The flexibility CH allows in choosing histories may seem to generate some difficulties. For example, a history in which the projector corresponding to (12.4.5) occurs, that is, a history in which Schrödinger's cat appears, although not forced on us, is still permissible. But, one might object, presumably one would want to have such a history forbidden. For if superposition has a physical counterpart, while one might swallow its application to a quantum particle, as in the CH equivalent of (12.8.1), one is unlikely to do the same with respect to a macroscopic object like a cat. However, as we saw, for CH the cat is neither dead nor alive, nor dead and alive; indeed, as this applies to any cat-property one might want to associate with $|\chi_i\rangle\langle\chi_i|$ in (12.4.5), it turns out that such a history has times when it cannot say anything about the cat. That is, in such history, at times when superposition occurs, any talk about the cat's properties is mere nonsense, and therefore *a fortiori* nothing funny is said about the cat.[16] This, however, seems to introduce a further difficulty, namely that in such a history there are times when the cat seems to vanish, as it were. But the cat, we should remember, is a macroscopic object: surely, one might insist following Einstein and Shrödinger, in any sensible

[16] I owe this point to an exchange with R. B. Griffiths.

description of the world Trickster must be in the box at all times with all its proper feline properties.

A related problem has to do with the fact that there is, at least at the macroscopic level, only one history, only one actual sequence of events, even if, of course, there are many possible histories. But CH cannot tell us which history is real, and which is, as it were, just a historical fable. True, to each history CH associates a probability, but even so, the theory is unable to tell us which shall emerge into reality. However, this criticism seems too harsh. After all, for CH the sequence of events in a history is, at least epistemologically, stochastic, which intimates that the selection of the actual history is, at least epistemologically, random. If so, it is unreasonable to expect CH to tell us more than the probability of each history and determine the causal chain, if it exists at all, that actualized one history rather than another. Even at the macroscopic level, because of the complexity of the causes involved, the best one can do is to provide probabilities when it comes to the histories of roulette's outcomes, leaving aside the issue of what caused one outcome rather than another.

Exercises

Exercise 12.1

Consider a fair coin tossed three times in a row. Determine the sample space. How many elements will the event algebra have?

Exercise 12.2

What is $3 \cap 4$ equivalent to, in terms of disjunction and negation? [Hint. Look at example 12.2.2]

Exercise 12.3

Using rules Q1 and Q2, show that the following equalities hold: a: $P \cap Q = Q \cap P$; b: $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$.

Exercise 12.4

Do $a = [\mathrm{X}\,+] \otimes [\mathrm{Z}\,-] \otimes [\mathrm{Y}\,-]$ and $b = [\mathrm{Y}\,+] \otimes [\mathrm{Z}\,+] \otimes [\mathrm{X}\,-]$ commute?

Exercise 12.5

1. Show that (12.5.5) can be written as $\Pr(a) = Tr\left(C_a \rho C_a^+\right)$. [Hint. Look at the rightmost factor in C_a. Then, remember that a projector is identical to its square.]
2. Show that (12.5.5) can be written as $\Pr(a) = \left\langle \Psi \middle| C_a^+ C_a \middle| \Psi \right\rangle$. [Hint. Start with the result of the previous exercise, cyclically rotate the argument of the trace operator, and then remember what $Tr(\rho A)$ is equal to.]
3. Prove that when the Hamiltonian is zero (and the system is conservative) the evolution operator becomes the identity operator.
4. Verify that in the example above the history probability we got agrees with the orthodox interpretation concerning the probability of obtaining 1, 1, and -1 were one

to perform the appropriate measurements in succession.

5. Is (4) true in general? [Hint. It turns out that $\Pr(a) = Tr\left(C_a C_a^{+}\right) = Tr\left(\widehat{C}_a \widehat{C}_a^{+}\right)$, where $\widehat{C}_a = \widehat{P}_n \widehat{P}_{n-1} \cdots \widehat{P}_0$, the product of the Heisenberg operators corresponding to the Schrödinger operators in (12.5.4), and this opens the way for the use of Wigner's formula. See appendix 4].

Exercise 12.6

In example 12.6.1 , show that

1. *a* and *c* are orthogonal.
2. $\langle C_a, C_b \rangle = 0$.
3. $\langle C_a, C_c \rangle \neq 0$.

Exercise 12.7

Show that if two histories *a* and *b* have orthogonal first or last projectors, then they are orthogonal.

Exercise 12.8

Show that the chain operators for h (12.8.7) and i (12.8.8) are not orthogonal. [Hint. Notice that the Hamiltonian is not zero, and therefore we use the evolution operators that need to be applied on density operators represented as ket-bra.]

Anwers to the Exercises

Exercise 12.1

The sample space is made up of 8 points, and therefore the event algebra has 2^8 elements.

Exercise 12.2

It is equivalent to $\sim(\sim 3 \cup \sim 4)$: it is not the case that a number different from 3 or 4 came up.

Exercise 12.3

a: Since P and Q must commute, by rule Q1 we get $PQ = QP$;

b: By rules Q1 and Q2, $P \cup (Q \cap R)$ is represented by $P + QR - PQR$.

$(P \cup Q) \cap (P \cup R)$ is represented by

$(P + Q - PQ)(P + R - PR) = P^2 + PR + P^2R + QP + QR + QPR - PQP - PQR + PQPR = P + QR - PQR$ once we keep in mind that the operators commute and a projector is the same as its square.

Exercise 12.4

Yes, because their second projectors are orthogonal.

Exercise 12.5

1. $Tr(C_a C_a^+) = Tr[P_n U(t_n, t_{n-1}) P_{n-1} \cdots U(t_1, t_0) P_0 P_0 U(t_0, t_1) \cdots P_{n-1} \cdots U(t_{n-1}, t_n) P_n]$. Since $P_0 P_0 = P_0 P_0 P_0 = P_0 \rho P_0$, we obtain $Tr(C_a C_a^+) = Tr(C_a \rho C_a^+) = \Pr(a)$

2. $Tr(C_a \rho C_a^+) = Tr(\rho C_a^+ C_a) = \langle C_a^+ C_a \rangle = \langle \Psi | C_a^+ C_a | \Psi \rangle$.

3. When the system is conservative, $U(t, t_0) = e^{\frac{-iH \cdot (t - t_0)}{\hbar}}$, and therefore when $H = 0$, $U(t, t_0) = I$.

4. Since the system is in state $|\uparrow_z\rangle$, $S_z = 1$ with probability 1; the second measurement

will return $S_x = 1$ with probability 1/2 and the state vector will collapse on $|\uparrow_x\rangle$; the third measurement will return $S_y = -1$ with probability 1/2. Hence, the probability of obtaining the three returns is $1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

5. Yes, because $Tr\left(\widehat{C}_a \widehat{C}_a^+\right) = Tr\left(\widehat{C}_a \rho \widehat{C}_a^+\right)$ is Wigner's formula.

Exercise 12.6

1. *a* and *c* are orthogonal because their second members are.

2. $C_b = [X-][Y+][Z-]$. Hence,

$\langle C_a, C_b \rangle = Tr\left(C_a^+ C_b\right) = Tr\left([Z+][Y+][X+][X-][Y+][Z-]\right) = 0$.

3. $\langle C_a, C_c \rangle = Tr\left(C_a^+ C_c\right) = Tr\left([Z+][Y+][X+][X+][Y-][Z+]\right)$

$= Tr\left([Z+][Y+][X+][Y-]\right) \neq 0$.

Exercise 12.7

Let $C_a = P_n U(t_n, t_{n-1}) P_{n-1} \cdots U(t_1, t_0) P_0$ and $C_b = Q_n U(t_n, t_{n-1}) Q_{n-1} \cdots U(t_1, t_0) Q_0$. Then,

$Tr\left(C_a^+ C_b\right) = Tr\left[P_0 U(t_0, t_1) \cdots P_{n-1} U(t_{n-1}, t_n) P_n Q_n U(t_n, t_{n-1}) Q_{n-1} \cdots U(t_1, t_0) Q_0\right]$. Hence, if $P_n Q_n = 0$ the result is immediate; if $Q_0 P_0 = 0$ the result is obtained by cyclical rotation.

Exercise 12.8

$C_h = |f\rangle\langle f| U(t_3, t_2) |c\rangle\langle c| U(t_2, t_1) |a\rangle\langle a|$, and $C_i = |f\rangle\langle f| U(t_3, t_2) |d\rangle\langle d| U(t_2, t_1) |a\rangle\langle a|$. Hence,

$Tr\left(C_h^+ C_i\right) = Tr\left[|a\rangle\langle a| U(t_1, t_2) |c\rangle\langle c| U(t_2, t_3) |f\rangle\langle f|f\rangle\langle f| U(t_3, t_2) |d\rangle\langle d| U(t_2, t_1) |a\rangle\langle a|\right]$. Since

$U(t_2, t_1) |a\rangle\langle a| = \frac{1}{\sqrt{2}}\left(|c\rangle + |d\rangle\right)\langle a|$, after a little algebra we get

$Tr\left(C_h^+ C_i\right) = \frac{1}{\sqrt{2}} \langle d|c\rangle Tr\left[|a\rangle\langle a| U(t_1, t_2) |c\rangle\langle c| U(t_2, t_3) |f\rangle\langle f|f\rangle\langle f| U(t_3, t_2) |d\rangle\langle a|\right]$. Continuing in the same way, we finally obtain $Tr\left(C_h^+ C_i\right) = \frac{1}{\sqrt{2}} \langle d | c \rangle \langle c | d \rangle \neq 0$, and therefore the two chain operators are not orthogonal.