Chapter 11

Measurement Problem II: the Modal Interpretation, the Epistemic Interpretation, the Relational Interpretation, Decoherence, and Wigner's Formula.

Previously, we have considered approaches to the measurement problem that deny either the universal validity principle, or the observer's reliability principle. We now turn to approaches that reject the remaining two principles or none at all. We start with the modal interpretation, which rejects EE. Then we turn to two views that reject the absolute state principle, namely, the epistemic and the relational interpretations. Finally, we consider an attempt at solving the measurement problem without denying any of the four principles that are necessary to its production, namely the decoherence approach. However, before tackling these views, we need some more theory.

11.1 The Projection Operator

The *Projection Operator* of a normalized vector $|\psi\rangle$ is $P_\psi = |\psi\rangle\langle\psi|$. If we apply P_ψ to a vector $|\Psi\rangle$, we obtain

$$P_\psi|\Psi\rangle = |\psi\rangle\langle\psi|\Psi\rangle = \lambda|\psi\rangle, \tag{11.1.1}$$

a multiple of $|\psi\rangle$ with the coefficient of proportionality λ being the scalar product $\langle\psi|\Psi\rangle$. The projector operator has two important properties, namely

$$P_\psi{}^2 = P_\psi \tag{11.1.2}$$

and

$$Tr(|\psi\rangle\langle\psi|) = 1. \tag{11.1.3}$$

Note that if $|\psi\rangle$ is a basis vector, then $P_\psi|\Psi\rangle$ is nothing but the ψ-component of $|\Psi\rangle$, the orthogonal projection of $|\Psi\rangle$ onto $|\psi\rangle$. Hence, if we project $|\Psi\rangle$ onto the basis vectors $|\psi_1\rangle,\ldots,|\psi_n\rangle$ we obtain the sum of $|\Psi\rangle$'s components, so that

$$|\Psi\rangle = \sum_i |\psi_i\rangle\langle\psi_i|\Psi\rangle = \sum_i P_{\psi_i}|\Psi\rangle. \tag{11.1.4}$$

Suppose now that $|\Psi\rangle = \sum_i c_i|\psi_i\rangle$. Then, as we know, $c_i = \langle\psi_i|\Psi\rangle$ and $c_i^* = \langle\Psi|\psi_i\rangle$.

Hence,

$$|c_i|^2 = \langle\Psi|\psi_i\rangle\langle\psi_i|\Psi\rangle = \langle P_i\rangle, \tag{11.1.5}$$

that is, the probability of getting the measurement return λ_i, the eigenvalue associated with the basis vector $|\psi_i\rangle$, is obtained by sandwiching the projector of that basis vector.

11.2 The Density Operator

Up to now, we have described a system by using its state vector. Such method has some drawbacks. For example, neither mixed states, nor subsystems of entangled states, although of obvious interest, can be represented by a state vector. The remedy is to introduce the notion of density operator, which allows the study of quantum systems uniformly.

Consider a mixture with probability $p_1, p_2, \ldots, p_n$ of being in the corresponding state $|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle$. As we know, if O is an observable, then

$$< O >= \sum_i p_i < O_{\Psi_i} >= \sum_i p_i\langle\Psi_i|O|\Psi_i\rangle. \tag{11.2.1}$$

We can now define the *density operator*

$$\rho = \sum_i p_i|\Psi_i\rangle\langle\Psi_i|.^{[1]} \tag{11.2.2}$$

Given that $P_i = |\Psi_i\rangle\langle\Psi_i|$ is a projector operator and therefore $Tr(P_i) = 1$, we may

[1] In other words, to construct the density operator, take the projector for each of the $|\Psi_i\rangle$, weigh it by the probability that the system is in state $|\Psi_i\rangle$, and sum all the thus weighted projectors. So, in a pure state system, $\rho = |\Psi\rangle\langle\Psi| = P_\Psi$, namely, Ψ's projector.

immediately infer that $Tr(\rho)=1$. Since $P_i^2 = P_i$, we have

$$Tr(P_iO) = Tr\left(P_i^2O\right) = Tr(P_iP_iO) = Tr(P_iOP_i), \tag{11.2.3}$$

where the rightmost member is obtained by cyclical rotation on the previous one. From (11.2.3) we obtain

$$Tr(P_iO) = Tr(|\Psi_i\rangle\langle\Psi_i|O|\Psi_i\rangle\langle\Psi_i|) = \langle\Psi_i|O|\Psi_i\rangle Tr(P_i), \tag{11.2.4}$$

and since $Tr(P_i)=1$ it follows that

$$Tr(P_iO) = \langle\Psi_i|O|\Psi_i\rangle, \tag{11.2.5}$$

By plugging (11.2.5) into (11.2.1), we obtain

$$< O >= \sum_i p_i Tr(|\Psi_i\rangle\langle\Psi_i|O). \tag{11.2.6}$$

Now let us note that $\rho O = \sum_i p_i|\Psi_i\rangle\langle\Psi_i|O$, and therefore

$$Tr(\rho O) = Tr\left(\sum_i p_i|\Psi_i\rangle\langle\Psi_i|O\right). \tag{11.2.7}$$

Since *Tr* is a linear operator, the trace of a sum is the sum of the traces, and consequently we can rewrite (11.2.7) as

$$Tr(\rho O) = \sum_i p_i Tr(|\Psi_i\rangle\langle\Psi_i|O). \tag{11.2.8}$$

Finally, by plugging (11.2.8) into (11.2.6) we obtain

$$< O >= Tr(\rho O), \tag{11.2.9}$$

which expresses *O*'s expectation value in terms of the density operator. In a pure state system such that $|\Psi\rangle = \sum_i c_i|\psi_i\rangle$, upon measuring *O* the probability of obtaining the eigenvalue λ_i is $\langle P_i\rangle$, as (11.1.4) shows. Hence, remembering that both P_i and the density operator ρ are Hermitian, by using (11.2.9) we obtain

$$|c_i|^2 = Tr(\rho P_i). \tag{11.2.10}$$

The density operator, then, provides a uniform procedure for calculating expectation values and the probabilities of individual returns for both pure and mixed state systems. More generally, all measurable quantities can be obtained by means of the density operator. Hence, while the state vector can only describe a pure state, a density operator can describe mixtures as well, thus providing a general way of representing any sort of system. In addition, while the same quantum state can be described by different state vectors, it can only be described by one density operator. A system is in a pure state if and only if the density operator reduces to a projection operator, and therefore $\rho^2 = \rho$.

Now let us construct the density operator for $|\Psi\rangle = \alpha|e_1 d_2\rangle + \beta|e_2 d_1\rangle$ the entangled state of particles 1 and 2, where $\{|e_1\rangle, |e_2\rangle\}$ is the basis for H_1, the space of particle 1, and $\{|d_1\rangle, |d_2\rangle\}$ is the basis for H_2, the space of particle 2. Keeping in mind that $|ab\rangle$ is shorthand for $|a\rangle \otimes |b\rangle$, we have

$$\rho = \left(\alpha|e_1\rangle \otimes |d_2\rangle + \beta|e_2\rangle \otimes |d_1\rangle\right)\left(\alpha^*\langle e_1| \otimes \langle d_2| + \beta^*\langle e_2| \otimes \langle d_1|\right), \tag{11.2.11}$$

that is,

$$\begin{aligned}\rho = {} & \alpha\alpha^*\left(|e_1\rangle \otimes |d_2\rangle\right)\left(\langle e_1| \otimes \langle d_2|\right) + \alpha\beta^*\left(|e_1\rangle \otimes |d_2\rangle\right)\left(\langle e_2| \otimes \langle d_1|\right) + \\ & \beta\alpha^*\left(|e_2\rangle \otimes |d_1\rangle\right)\left(\langle e_1| \otimes \langle d_2|\right) + \beta\beta^*\left(|e_2\rangle \otimes |d_1\rangle\right)\left(\langle e_2| \otimes \langle d_1|\right).\end{aligned} \tag{11.2.12}$$

Since $(A \otimes B)(C \otimes D) = AC \otimes BD$, we obtain

$$\begin{aligned}\rho = {} & \alpha\alpha^*\left(|e_1\rangle\langle e_1| \otimes |d_2\rangle\langle d_2|\right) + \alpha\beta^*\left(|e_1\rangle\langle e_2| \otimes |d_2\rangle\langle d_1|\right) + \\ & \beta\alpha^*\left(|e_2\rangle\langle e_1| \otimes |d_1\rangle\langle d_2|\right) + \beta\beta^*\left(|e_2\rangle\langle e_2| \otimes |d_1\rangle\langle d_1|\right).\end{aligned} \tag{11.2.13}$$

11.3 Reduced Operators

Suppose that particles 1 and 2 have become entangled, and that we want to make predictions about observable O of particle 1. Obviously, one way is appropriately to apply ρ, the density operator for system 1+2, to O's extension. However, it is also possible to construct a new density operator ρ_1, the reduced density operator of 1, which operates in

H_1, and therefore can be applied to O directly. The matrix element $o_{i,j}$ of ρ_1 is obtained by performing a *partial trace* on ρ with respect to 2:

$$o_{i,j} = \sum_m \left(\langle e_i | \langle d_m |\right) \rho \left(| e_j \rangle | d_m \rangle\right), \tag{11.3.1}$$

so that $\rho_1 = Tr_2(\rho)$. In other words, to obtain ρ_1, we apply the operator Tr to the parts of ρ containing basis vectors from 2. Hence, if (11.2.13) gives the density operator for the system 1+2, then

$$\rho_1 = \alpha\alpha^* \left(|e_1\rangle\langle e_1| \otimes Tr|d_2\rangle\langle d_2|\right) + \alpha\beta^* \left(|e_1\rangle\langle e_2| \otimes Tr|d_2\rangle\langle d_1|\right) +$$
$$\beta\alpha^* \left(|e_2\rangle\langle e_1| \otimes Tr|d_1\rangle\langle d_2|\right) + \beta\beta^* \left(|e_2\rangle\langle e_2| \otimes Tr|d_1\rangle\langle d_1|\right),$$

(11.3.2)

and remembering that $Tr\left(|A\rangle\langle B|\right) = \langle B|A\rangle$ we obtain

$$\rho_1 = \alpha\alpha^* |e_1\rangle\langle e_1| + \beta\beta^* |e_2\rangle\langle e_2|. \tag{11.3.3}$$

It can be shown that ρ_1 is in fact a density operator and that it contains the same information about 1 as the state vector of the compound system.[2]

11.4 The Modal Interpretation

Under the label "modal interpretation," one can group a rather varied assortment of interpretations of quantum mechanics that share the rejection of EE. The field of modal interpretation is far from settled, with several authors holding rather different views, and consequently we shall restrict ourselves to very general considerations.[3]

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[2] However, there is no Hamiltonian operator relating to, say, 1 alone, enabling us to obtain the evolution equation for ρ_1. By contrast, there is no such problem for the density operator of the compound.

[3] For a general account of the modal interpretation, see Vermaas, P. E., (1999).

The basic idea is that the standard quantum state never collapses and that its job is to provide the probabilities, interpreted in terms of ignorance, that the system possesses one among a certain set of properties. The problem, of course, is to come up with such a set of properties without falling foul of KS. To this effect, typically, one distinguishes the *value state* from the *dynamical state* of a system. The latter is the standard, but non-collapsable, quantum state. The value state, an entity that does not exist in the standard interpretation, describes the properties of the system.

One way to understand how this can be made to work is by appealing to Schmidt's theorem (the bi-orthogonal decomposition theorem). Consider a system S made up of two disjoint subsystems S_1 and S_2 and whose Hilbert space is $H = H_1 \otimes H_2$. Then, for every state vector $|\Psi\rangle$ in H it is the case that

$$|\Psi\rangle = \sum_i c_i |\psi_i\rangle \otimes |\phi_i\rangle, \tag{11.4.1}$$

where the set of $|\psi\rangle$'s is an orthonormal basis (the Schmidt basis) in H_1, the set of $|\phi\rangle$'s an orthonormal basis (the Schmidt basis) in H_2 and $\sum_i |c_i|^2 = 1$.[4] For example, the singlet configuration (8.4.1) takes the form of a bi-orthogonal decomposition. (Note that the vector expansion has only two addenda instead of four). It turns out that the decomposition is unique if and only if no two $|c_i|^2$ are equal.[5] For example, the singlet configuration

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[4] Given (11.4.1), one can easily obtain the reduced density operators $\rho_1 = \sum_i |c_i|^2 |\psi_i\rangle\langle\psi_i|$ and $\rho_2 = \sum_i |c_i|^2 |\phi_i\rangle\langle\phi_i|$.

[5] The theorem holds only for a two-component system, although each subsystem can be made up of further subsystems. When the various $|c_i|^2$ are not all different the decomposition is degenerate and things get more complicated. Still, modal interpretations

(8.4.1) is not unique because $|c_1|^2 = |c_2|^2$, and in fact, we saw that (8.4.1) enjoys rotational invariance. When the decomposition is unique, the state of S picks out a unique basis $\{|\psi_1\rangle,...,|\psi_n\rangle\}$, and therefore an observable O, for S_1. The same is true of S_2, where an observable Q is picked by the state of S. Then, each of the $|\psi\rangle$'s is a value state for S_1 and each of the $|\phi\rangle$'s a value state for S_2.

At this point, quantum mechanical computation takes over. The dynamical state $|\Psi\rangle$ gives a probability $|c_i|^2$ that S_1 has a value state $|\psi_i\rangle$ and S_2 value state $|\phi_i\rangle$. The adoption of a modification of the eigenstate to eigenvalue link, namely, the *value*-eigenstate to eigenvalue link, guarantees that O and Q have the appropriate eigenvalues. In particular, when, say, S_2 is a measuring device and S_1 the observed system, the entanglement between S_1 and S_2 brought about by TDSE is in fact a bi-orthogonal decomposition. The rejection of EE allows one to say that pointers point even when they are represented by a superposition, thus eliminating the pure state problem. At the same time, it becomes possible to say that O has a determinate value even before measurement even if we do not know it. The strictures imposed by KS are satisfied by requiring that S_1 may not possess all possible properties all the times. Instead, S_1 is only ascribed O and all the properties obtainable from it by manipulation in terms of the logical connectives "not", "or", "and."[6] Other properties are taken to be undefined. We may call the view just described "the bi-

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can be extended to cover such cases. For simplicity, we shall stick to non-degenerate cases.

[6] These logical connectives are mapped onto a subset of the relevant Hilbert space. There is not a general agreement on how to do this. We shall see one possible way of doing it when we study the Consistent Histories Interpretation, which constructs a Boolean algebra (section 12.3).

orthogonal interpretation." Its predictions are identical to those of standard quantum mechanics.

The bi-orthogonal interpretation suffers from two main problems. First, the system $S = S_1 + S_2$ must be in a pure state. Second, the fact that Schmidt's theorem applies only to two-component systems entails that S_1 has property *O* only from the point of view of S_2. In addition, if we couple S_1 not with S_2 but with, say, S_3, the bi-orthogonal decomposition of the state vector for this new compound will not choose the set of $|\psi\rangle$'s as an orthonormal basis in H_1, and this reinforces the point that a system's properties (not just their values) are not intrinsic but relational since they exist only in connection to other systems, a fact often referred to as "perspectivalism." [7] In addition, when one of the two systems is a measurement device, perspectivalism produces a type of environmental contextuality that some find problematic because it limits one's ability to examine correlations. For example, in the singlet configuration, where *a* and *b* are the two entangled particles and *A* and *B* are the devices measuring them, the total system is *abAB*. In terms of the biorthogonal interpretation, the system can be divided into *A* and *abB* or into *B* and *abA*, and the respective pointer positions cannot be correlated because they correspond to different perspectives.

However, to a certain degree these two problems can be diminished by altering the bi-orthogonal version. For, it turns out that when (11.4.1) applies, the bi-orthogonal decomposition is unique, and

$$W = \sum_i c_i |\chi_i\rangle\langle\chi_i| \qquad (11.4.2)$$

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[7] Hence, perspectivalism, as it is here understood, is more radical than property contextualism. For the latter, not the properties but their values are context dependent; for the former, the very existence of a property is contextual.

is the reduced density operator for subsystem S_1, then $|\psi_i\rangle = |\chi_i\rangle$. In other words, in the Schmidt basis for S_1 of system $S = S_1 + S_2$, the matrix of *W* is diagonalized. Hence, instead of using the bi-orthogonal decomposition theorem, one may use the familiar reduced density operators in order to determine S_1's properties. One may then generalize this result into what is called "the spectral interpretation" and claim that the value states of any system are provided by the spectral resolution of its density operator. This allows the consideration of subsystems $S_1,\ldots,S_n$ that are not part of a composite *S* in a pure state, and therefore it solves the first problem. Moreover, it allows the correlation of the properties of the measuring devices *A* and *B* in the EPR case described above, thus diminishing the role of perspectivalism.

Nevertheless, some degree of perspectivalism remains. For the spectral interpretation, like the bi-orthogonal interpretation, allows the correlation of properties of subsystems $S_1,\ldots,S_n$ only if such subsystems are disjoint. Since if a system *S* has more than two subsystems then it can be partitioned into disjoint subsystems in more than one way, correlations can be established only among those properties belonging to subsystems involved in one and the same partition. For example, suppose that $S = S_1 + S_2 + S_3$. Obviously, one can partition S into S_1, S_2, S_3, or into $(S_1 S_2), S_3$, and so on. Consequently, any correlation of properties depends on a specific partition, and therefore perspectivalism remains.

The obvious solution is to claim that there is one and only one privileged partition of a complex system. In other words, one may claim that there are disjoint elementary systems that are somehow fundamental in nature and that complex systems are built out of them. This is the basic idea behind the atomic interpretation. In this view, the properties of the fundamental (atomic) systems are obtained according to the spectral interpretation. However, the property-projectors of a compound (molecular) system are obtained differently: they are the tensor product of the property-projectors of the component atomic systems.

In spite of their ingenuity, it remains unclear whether modal interpretations can successfully cope with measurement. One can show that if the modal interpretations are correct there are some possible measurements that have no outcome. Although no doubt serious, this may not be a fatal drawback, as long as the measurements in question are outlandish and utterly unrealistic. The problem is that if the measurement device must be described in an infinite dimensional Hilbert space (as it is the case with a pointer allowing a continuous set of readings), then these interpretations fail to account for measurement outcomes. This casts doubts that modal interpretations are empirically adequate.

11.5 The Epistemic Interpretation

Some proposals to solve the measurement problem are based on the rejection of the absolute state principle, the view that the state vector is unqualifiedly about the physical state of a system. The most radical of these is what may be dubbed the "Epistemic Interpretation", according to which the state vector does not describe a physical system but our knowledge of its possible experimental behaviour, and its collapse at measurement merely refers to a change in such knowledge.[8] In other words, the temporal evolution of the wavefunction does not describe the evolution of a physical system but the change in the

[8] Fuchs, C. A. and Peres, A., (2000), if I understand them correctly. See also their reply to comments in the same journal, (September 2000), 11-14 and 90. Similar views are held by Peierls (Peierls, R. E., (1991)), and Zeilinger (Zeilinger, A., (1999)). Later in his career, Heisenberg claimed that the state function contains both objective and subjective elements, the former ones associated with the potentialities present in the system, and the latter ones reducible to the individual's knowledge of the system; the function's collapse is just the result of a change in knowledge (Heisenberg, W., (1971): 53-4).

probabilities of possible experimental returns given the observer's knowledge. So, Schrödinger's cat is not, as it were, half dead and half alive; rather, the state of superposition simply represents our imperfect knowledge of the cat's state. Upon observation, our knowledge, but not the cat, abruptly changes, and this is represented by the wave function collapse. There is no measurement problem because TDSE and collapse describe two different and well-defined epistemic states, one before and one after measurement.

The idea that the state vector does not really express a physical state can be supported by considering interaction-free measurements. A photon represented by the wave packet $|a\rangle$ is traveling in a Mach-Zehnder interferometer along path *a* toward a beam splitter with equal reflection and transmission indexes, so that the probability the photon is reflected is the same as the probability it is transmitted, namely, 1/2 (Fig. 1).

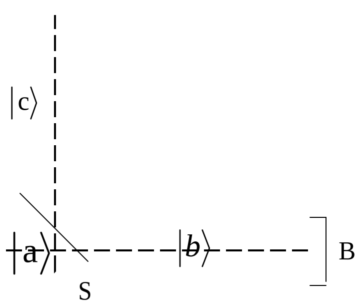

Figure 1

Suppose a transmitted photon moving along path *b* is in state $|b\rangle$ and a reflected photon moving along path *c* is in state $|c\rangle$. On path *b* there is a detector B in state $|B\rangle$ when not activated by a photon, and in state $|B^*\rangle$ when activated by one; on path *c* there is nothing.

Let us keep our attention on B. If after a photon goes through the beam splitter detector B is not activated we can infer that the photon is on path *c*. For, at the beginning, the state of the system photon-plus-detector is $|a\rangle \otimes |B\rangle$; once the photon has gone through the beam splitter, the state becomes

$$\sqrt{\frac{1}{2}}\left[|b\rangle \otimes |B\rangle + |c\rangle \otimes |B\rangle\right], \qquad (11.5.1)$$

and eventually

$$\sqrt{\frac{1}{2}}\left[|b\rangle \otimes |B^*\rangle + |c\rangle \otimes |B\rangle\right]. \qquad (11.5.2)$$

Since the detector is not activated, the system's state has collapsed onto $|c\rangle|B\rangle$, and therefore the photon is on path *c*. Note that collapse has occurred even if, apparently, since the photon did not activate the detector, nothing occurred physically; in other words, nothing interacted with the photon.[9] Since this intimates, albeit it does not prove, that collapse is not a physical event, the epistemological interpretation seems quite reasonable.

The basic problem for the epistemological interpretation is to avoid falling foul of the distinction between pure and mixed states. Considering the cat definitely dead or definitely alive just before we open the box leads to the wrong prediction by ignoring the interference between the two states. In other words, the epistemological move cannot work without other assumptions. Fuchs and Peres simply claim that we can say nothing about the cat in-between observations. However, the cat is a macroscopic object, and it seems preposterous to hold that we cannot even say that the cat is either alive or dead when we do not look at it. Of course, it may turn out that we cannot say anything about the cat, but a lot of fancy footwork would be required to make such a view compelling.

One might argue that the success of quantum mechanics strongly intimates that it

[9] For a nice discussion of interaction free measurement, see De Weerd, J., (2002).

correctly describes physical reality. However, Fuchs and Peres claim, such a description is not required for success; after all, probability theory gives reliable results without describing the physics of the roulette wheel. Of course, many, from Einstein to Schrödinger and Bell, have thought of this view of quantum mechanics as unduly restrictive. However, according to Fuchs and Peres, such an attitude is unjustified because the task of science is just to provide a compact description of physical experience and to predict experimental outcomes. That in the case of classical physics we have been able to produce a model of a reality independent of our experiments is nice but not strictly necessary. In short, they adopt a strict positivist position on the role of science.

If the state vector is not about physical systems but about our knowledge of them, then TDSE depicts some sort of normative psychological dynamics, and, in a way, quantum physics becomes some sort of theory of conditional probabilities dealing with our legitimate expectations about measurement returns rather than with our actual expectations about measurement returns.[10] In other words, it not a descriptive but a normative discipline, and in this respect not like psychology, which tells us how we think, but like logic, which tells us how we ought to think. Even so, however, the transformation of a physical theory from discipline that primarily describes the working of nature and secondarily tells us how we should think about nature to a discipline that disregards the former task and aims merely to the latter is radical and to many utterly unpalatable.

11.6 The Relational Interpretation

[10] As Einstein remarked to Schrödinger, this is "the Born interpretation, which most theorists today probably share. But then the laws of nature that one can formulate do not apply to the change with time of something that exists, but rather to the time variation of the content of our legitimate expectations." (Einstein to Schödinger, August 9, 1939, in Przibram, K., (ed.) (1967): 35).

Another attempt at getting around the measurement problem by rejecting the view that the state vector is unqualifiedly about the physical state of a system is the Relational Interpretation (RI). In classical mechanics, velocities make sense only in relation to a frame of reference (an observer) and the velocities of an object with respect to two different frames of reference need not agree. However, other physical quantities such as length or duration are invariant in the sense that their values are the same no matter which frame of reference we choose. One might view part of the history of 20th century physics as a reshuffling of which physical properties are invariant and which are not. According to Bohr, as Special Relativity shows that length and duration, invariant in classical mechanics, are in reality not invariant and therefore their values are only ascribable relative to a reference frame, so quantum mechanics shows that the values of quantum dynamical properties are ascribable only relative to an experimental setup. That not only the values of quantum dynamical properties but also those of quantum states, quantum relations, and measurement for a physical system S are relative to another physical system O (the observer system) is the basic idea of RI.[11] The observer system can be any system, a micro-system, a macro-system, an apparatus, or an experimenter, so that "observer system" need not, although it might, carry any connotation of consciousness.

Consider the following standard quantum mechanical account. Let S be a spin-half particle which at time t_0 is in a state represented by

$$\left|\Psi_0\right\rangle = a\left|\uparrow_z\right\rangle + b\left|\downarrow_z\right\rangle. \tag{11.6.1}$$

Suppose we measure S_z and the measurement consists in the interaction between S and another system whatsoever O, which for simplicity we assume to be a SGZ. Suppose also that at t_1 the measurement result is $S_z = 1$, in which case collapse has taken place and

[11] Here we follow Rovelli, C., (1996).

$$|\Psi_1\rangle = |\uparrow_z\rangle. \qquad (11.6.2)$$

Let E be the sequence of events from t_0 to t_1. Then, from O's point of view, S went from a state represented by (11.6.1) to one represented by (11.6.2), acquiring $S_z = 1$ at t_1.

Now let us introduce a new system O′ which is not a subsystem of either S or O, and which describes E by considering the compound system S+O without interacting with it (without measuring it). At t_0, the state of S+O is represented by

$$|\Phi_0\rangle = \left(a|\uparrow_z\rangle + b|\downarrow_z\rangle\right) \otimes |\chi_0\rangle, \qquad (11.6.3)$$

where $|\chi_0\rangle$ is O's initial (ready) state. At time t_1, (11.6.3) has evolved into

$$|\Phi_0\rangle = a|\uparrow_z\rangle \otimes |\chi_+\rangle + b|\downarrow_z\rangle \otimes |\chi_-\rangle, \qquad (11.6.4)$$

which exhibits the correlation between S's and O's variables. Hence, from the point of view of O′, E is described by (11.6.3)-(11.6.4).

Formulas (11.6.1)-(11.6.2) and (11.6.3)-(11.6.4) offer two different accounts of E. For example, barring the case when $a = 0$ or $b = 0$, (11.6.4) contains no information about the result of the measurement. Worse, according to the account from O's point of view, at t_1, $S_z = 1$ and S is in the state represented by $|\uparrow_z\rangle$. By contrast, according to the account from the point of view of O′, at t_1, the system S is in a state of superposition, and S_z does not even exist, at least if one adopts EE. Both accounts are correct (their conjunction is a version of the measurement problem), and yet they are incompatible. Hence, *if* we add the state completeness principle (as RI does) we are led to the conclusion that S's quantum states, the values of S_z, and therefore measurement outcomes, are not absolute, but relative to the observer system. That is, relative to O, (11.6.1)-(11.6.2) is true; relative to O′, (11.6.3)-(11.6.4) is. Therefore, according to RI there is no conflict between the two accounts.

At this point, one might object that only one of the two accounts is true, and therefore there is no reason at all to accept RI. For, if O is the right sort of system, for example, a measuring device or a mind, then collapse takes place (absolutely) at t_1, and therefore (11.6.1)-(11.6.2) is true but (11.6.3)-(11.6.4) false. By contrast, if O is not of the right sort, then there is no collapse, and therefore (11.6.3)-(11.6.4) is true but (11.6.1)-(11.6.2) is not. On the face of it, this objection is very powerful. After all, an orthodox theorist might continue, in the above example, O is a SGZ and there *is* collapse. In other words, the assumption that there are special collapse-inducing systems explains why collapse occurs and eliminates the need for relationalism, and therefore we ought to make that assumption.[12] However, proponents of RI disagree. They do not believe that postulating the existence of special collapse inducing systems is the best available explanation of collapse since they think they can provide a better one. In addition, they reasonably hold that *all* systems are equivalent in the sense of being in principle describable in quantum mechanical terms and in having the capacity to become entangled with other systems, thus generating correlations of the sort described by (11.6.4). There are no privileged systems that induce collapse absolutely, that is, relative to all possible observers.

11.7 Measurement According to RI

As the notion of quantum state is relational (a system is in a certain state only in relation to another system), so is that of collapse: collapse may occur with respect to a

[12] How much the claim that, say, the mind has a collapsing capacity explains anything is difficult to say. Obviously, it explains something: it is the mind and not planet Venus that produces the collapse. However, one is reminded of the 'dormitive power' of opium as the explanation why some chap has fallen asleep after taking an opium, the stock example used by early modern philosophers to ridicule alleged Aristotelian-scholastic explanations of phenomena.

system O but not with respect to another system O′. Here is why. Unitary evolution of a system requires that the system be isolated, that is, that all the relevant Hamiltonians be expressed in the Hamiltonian operator entering TDSE. Since TDSE is always written from the point of view of an observer system, this requirement can be fulfilled only if the relevant information is available from the perspective of that observer. Suppose the observer system is O in the example above. In order to measure S_z is must interact with S. Hence, from the point of view of O, the system, namely S+O, is isolated only if O contains information about the interaction Hamiltonian, and ultimately about S's Hamiltonian and its own. While O contains, or at least may contain, adequate information about S's Hamiltonian, it cannot contain adequate information about its own. The reason is that a system O has information about a system S only if there is a correlation between S's and O's variables. For example, a pointer has information about a physical quantity Q if the pointer's positions and Q's values are correlated. Now, Rovelli claims, it makes no sense to be correlated with oneself (Rovelli, C., (1996): 1666). One might object to such a claim, and perhaps argue that self-conscious systems have the capacity to know their mental states by introspecting. However, aside from the dubious psychology involved in the previous claim, there is some reasonable logical evidence that no system can be so correlated as to have total information about itself.[13] We may interpret this as entailing that no physical system can have exact information about its own Hamiltonian. In short, from the perspective of O, the system O+S is not isolated and the Hamiltonian entering TDSE is far from complete. The result is that in relation to O, S's state vector collapses and its physical state undergoes an abrupt change.

[13] Dalla Chiara, M.L., (1977); Breuer, T., (1995).

By contrast, O′ may, and in fact does, contain adequate information about the Hamiltonians involved in S+O, and consequently from its perspective the state development of S+O is unitary. There is no conflict between the two types of development because state systems are relational by nature. In short, there is no mystery in collapse per se, although why the collapse is onto one eigenvector rather than another, that is, why one gets the measurement return one gets, does remain completely mysterious. There is another aspect of measurement that RI can clarify, namely, when measurement takes place. As we saw when discussing von Neumann' s view, Rovelli introduces an operator *M* on the space of S+O capable of telling us when the correlation between measured variable and pointer position occurs.[14] Of course, the whole exercise makes sense only from the perspective of O′, but this, according to RI, is inevitable. An analogous point is also evident in the RI treatment of EPR-like situations. The two entangled particles have their anti-correlated properties only with respect to an observer O, located in the proximity of particle *a*, or in relation to an observer O′, located in the proximity of particle *b*. Any conclusion one might want to derive will also be relative to one of the two observers.[15]

If RI is correct, there cannot be any quantum mechanical, observer-independent, universal description of a system. To paraphrase Rovelli, the reason is that physics is only about the relative information systems have regarding each other, and this information is all one can say about the world (Rovelli, C., (1996): 1655).[16] Consequently, contrary to

[14] As we noted, when *M* has the values it has, is a matter of debate.

[15] For details, see Laudisia, F., (2001).

[16] Hence, RI and the perspectival versions of the Modal Interpretation are quite close. By contrast, Everett's relative state formulation is not, as relative states are so not in relation to another system but in relation to its states. RI is about relations among systems, not states.

Everett, there is no quantum description of the universe as a whole because by hypothesis there is no observer system that is not a subsystem of the universe. Similarly, contrary to Einstein, the observer, even if conscious, cannot discretely fade in the background. At this point, one might object that if RI is right, physics is unable to tell us how things *really* are because its accounts are bound to be relational, but RI rejects this line of argument. There is no privileged observer, something like Newton's absolute space and time in relation to which things really are one way or another. In fact, according to Rovelli an appeal to an observer-independent description of a system is meaningless. However, in practice it is helpful to agree on a class of privileged systems (macroscopic systems we are able to use as measurement apparatuses) in relation to which quantum phenomena are studied, and consequently always discuss collapse only relative to them. Still, this should not obscure the fact that such an agreement merely reflects our human idiosyncrasies, as all systems are equivalent.

11.8 Just Density Operators?

There have been attempts at getting around the measurement problem *without* denying the universal validity principle, the observer's reliability principle, the eigenvalue-only-if-eigenstate principle, or the absolute state principle. As we saw, one of the distressing features of how quantum measurement is standardly handled is the breach in the continuous and linear evolution of the state vector caused by collapse. Suppose, however, that this breach is just the result of a certain type of mathematical approach centered on the notion of state vector; in other words, suppose that it is a mathematical artifact, as it were. Then, it might be possible to avoid it altogether by approaching measurement from a different mathematical perspective. In fact, one *can* 'do' quantum mechanics by using only density operators, without appealing to the state vector. To see how this works, we need to look at the temporal evolution of the density operator. Let us start by noting that the derivative of an operator is just the derivative of each of the elements of the corresponding matrix, and that the derivation rules are identical to those for functions as long as one does not change the order of the operators.[17] Now consider the density operator $\rho = |\Psi\rangle\langle\Psi|$ of a pure state system $|\Psi\rangle$. Then, by applying the rule for the derivative of a product

$$\frac{d}{dt}\rho = \frac{d}{dt}\left(|\Psi\rangle\langle\Psi|\right) = \left(\frac{d}{dt}|\Psi\rangle\right)\langle\Psi| + |\Psi\rangle\left(\frac{d}{dt}\langle\Psi|\right). \qquad (11.8.1)$$

Now as we know, TDSE can be written as

$$i\hbar\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle, \qquad (11.8.2)$$

where H is the system's Hamiltonian. In addition, using the rule for the manipulation of Dirac formulae, the complex conjugate of TDSE is

$$-i\hbar\frac{d}{dt}\langle\Psi| = \langle\Psi|H, \qquad (11.8.3)$$

since H is Hermitian and therefore equal to its adjoint.

By plugging (11.8.2) and (11.8.3) into (11.8.1), we obtain

$$\frac{d}{dt}\rho = \frac{1}{i\hbar}H|\Psi\rangle\langle\Psi| - \frac{1}{i\hbar}|\Psi\rangle\langle\Psi|H, \qquad (11.8.4)$$

that is,

$$\frac{d}{dt}\rho = \frac{1}{i\hbar}[H,\rho], \qquad (11.8.5)$$

the equation ruling the time evolution of the density operator for a pure state. Equation (11.8.5), in effect, plays on the density operator the role that TDSE plays on state vectors. The conservation of probability is given by the fact that at all times $Tr(\rho)=1$. So, it is possible to take density operators as containing all that we can know about quantum systems and use them together with (11.8.5) instead of state vectors and TDSE.

At this point, one might argue that collapse can now then be taken out of the picture

[17] For an introduction to derivatives, see appendix 1.

and replaced by the trace operator *Tr*, which is linear. The problem is that in a density operator the interference terms have not disappeared and are visible in the corresponding matrix as non-diagonal terms. However, once the compound system made up of measured system plus measuring apparatus comes into play, one uses the reduced density operator for the object system according to the procedure described before, and then something remarkable, decoherence, takes place.

11.9 Decoherence

Decoherence is a process whereby a system S correlated with a system E appears to be in a mixed state to someone measuring S alone. Before we see how this might happen, let us remember that the most obvious difference between the pure state $|\Psi\rangle = \alpha|e_1d_2\rangle + \beta|e_2d_1\rangle$ and the corresponding mixed state $W = |\alpha|^2|e_1d_2\rangle + |\beta|^2|e_2d_1\rangle$ is given by the fact that the former involves superposition interference and the latter does not. This is particularly clear at the level of density operators. As we know, for the system in a pure state the density operator is

$$\rho_\Psi = \alpha\alpha^*\left(|e_1\rangle\langle e_1|\otimes|d_2\rangle\langle d_2|\right) + \beta\beta^*\left(|e_2\rangle\langle e_2|\otimes|d_1\rangle\langle d_1|\right) + \alpha\beta^*\left(|e_1\rangle\langle e_2|\otimes|d_2\rangle\langle d_1|\right) + \beta\alpha^*\left(|e_2\rangle\langle e_1|\otimes|d_1\rangle\langle d_2|\right), \tag{11.9.1}$$

and the corresponding matrix is

$$\rho_\Psi = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}. \tag{11.9.2}$$

By contrast, for the mixed state,

$$\rho_W = |\alpha|^2\left(|e_1\rangle\langle e_1|\otimes|d_2\rangle\langle d_2|\right) + |\beta|^2\left(|e_2\rangle\langle e_2|\otimes|d_1\rangle\langle d_1|\right), \tag{11.9.3}$$

and the corresponding density matrix is

$$\rho_W = \begin{pmatrix} \alpha\alpha^* & 0 \\ 0 & \beta\beta^* \end{pmatrix}. \tag{11.9.4}$$

In other words, the most obvious difference is given by the cross terms (present in the pure state and absent in the mixed state) or, in terms of matrices, by the off-diagonal elements (different from zero in the pure state and equal to zero in the mixed state).[18]

Decoherence is a process by which a system in state $|\Psi\rangle$ interacts with the environment in such a way that the cross terms of its density operator (the off-diagonal elements of its density matrix) become practically indistinguishable from zero. In other words, ρ_Ψ and ρ_W effectively coincide in the sense that the expectation values of any operator calculated using the former are empirically identical to those calculated using the latter. Then, it seems reasonable to think that although $|\Psi\rangle$ is a pure state, the system in fact behaves as if it were in a mixed state when we measure it. To see how this might come about, let us look at the following example, due to Laloë (Laloë, F., (2001)).

EXAMPLE 11.9.1

Consider a system N of 2*n* atoms that have gone trough a SGD that has correlated their spin directions with positions in space so that the state vector is

$$|\Psi\rangle = \alpha\left(|1+\rangle \otimes \ldots \otimes |n+\rangle\right) + \beta\left(|1-\rangle \otimes \ldots \otimes |n-\rangle\right) = \alpha|A\rangle + \beta|B\rangle, \qquad (11.9.5)$$

where $|i+\rangle$ is the state of atom *i* after it exited the spin-up side of the SGD, and analogously for $|i-\rangle$. Suppose now that a photon K interacts with the atoms and is therefore scattered into state $|k+\rangle$ if it interacts with atoms in state $|i+\rangle$ and into state $|k-\rangle$ if it interacts with atoms in state $|i-\rangle$. Then the state of the new system N+K is

$$|\Psi'\rangle = \alpha|A\rangle \otimes |k+\rangle + \beta|B\rangle \otimes |k-\rangle, \qquad (11.9.6)$$

and the reduced density operator for N is

$$\rho_N = \alpha\alpha^*|A\rangle\langle A| + \beta\beta^*|B\rangle\langle B| + \alpha\beta^*\left(|A\rangle\langle B| \otimes \langle k-|k+\rangle\right) + \beta\alpha^*\left(|B\rangle\langle A| \otimes \langle k+|k-\rangle\right). \qquad (11.9.7)$$

If the distance between the two sides of the SGD is macroscopic, $|k+\rangle$ and $|k-\rangle$ will be

[18] Whether this is the *only* difference, however, is another question, as we shall see.

orthogonal or nearly so, with the result that the interference members of ρ_N (the off-diagonal elements of the corresponding density matrix) will be zero or nearly so as well. In addition, multiple scattering events will make the interference members tend exponentially to zero.

The example shows that in general, when atomic states are located at different places as they must be in macroscopic measuring devices, the interaction with environmental particles will destroy their coherence. If we couple this with the fact that macroscopic objects are awash in particles and that decoherence time for macroscopic objects is phenomenally fast, we have an explanation why interference effects are effectively absent in the macro-world but in extreme circumstances in which environmental influence is very greatly reduced. [19] If superposition were just interference, then decoherence would solve the measurement problem by explaining why measuring devices, or Schrödinger's cat, are never found in a state of superposition. That is, the superposition members of their density operators become so close to zero so quickly that interference, although present, as far as we are concerned never appears. In other words, macroscopic devices, when allowed to interact with the environment, behave as if they were in a mixed

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[19] A dust particle of radius of about $10^{-3} cm$ in a vacuum containing only microwave background radiation has a decoherence time of about $10^{-6} s$; the decoherence time for the same particle in normal air temperature and pressure drops to about $10^{-36} s$. We can get a sense of the magnitudes involved by noting that the age of the universe is about $10^{17} s$ (Home, D., (1997): 155). Still, by almost eliminating environmental interference it has been possible to create a case of macroscopic quantum tunneling by using a Superconducting Quantum Interference Devices (SQUID). For more, see Greenstein, G., and Zajonc, A., (1997): 171-77. Decoherence at the micro-level is much slower, which explains why interference plays such a large role.

state even when they are in a pure state.

However, there is more to superposition than interference. To see why, let us distinguish *proper* and *improper* mixtures. All the mixtures we have considered up to now are proper because the state of the system is given by just one of the summands in the statistical mixture, although we do not know which. For example, in (11.9.3) the system is *either* in state $(|e_1\rangle\langle e_1|\otimes|d_2\rangle\langle d_2|)$ *or* in state $(|e_2\rangle\langle e_2|\otimes|d_1\rangle\langle d_1|)$: the two states are mutually exclusive. However, in (11.9.1) the system's state is still a superposition, albeit with fewer effective members than before, presumably a combination of actually simultaneously present components $(|e_1\rangle\langle e_1|\otimes|d_2\rangle\langle d_2|)$, $(|e_2\rangle\langle e_2|\otimes|d_1\rangle\langle d_1|)$, $(|e_1\rangle\langle e_2|\otimes|d_2\rangle\langle d_1|)$, and $(|e_2\rangle\langle e_1|\otimes|d_1\rangle\langle d_2|)$: as far as one can see, the four states are not mutually exclusive because they somehow combine to make a quantum state that seems to defy interpretation. When decoherence makes the last two vanish we obtain an improper mixture because the vanishing act does not alter the fact that the first two are not only quantum mechanically compatible, but somehow "co-present" (and nobody understand what this really amount to). As Bell noted, to pretend otherwise involves the fallacy of converting an "and" into an "or" (Home, 84-6). One might disagree with Bell's contention that we are really dealing with an "and", but for sure we are not dealing with an "or". In short, decoherence cannot show why *any* determinate result comes about. As Laloe puts it: "During decoherence, the off-diagonal elements of the density matrix vanish (decoherence), while in a second step all diagonal elements but one should vanish (emergence of a single result)" (Laloë, F., (2001): 677). Differently put, although it need not explain why we got *this* return, any solution to the measurement problem has to explain why we got *one* return. Consequently, decoherence does not reconcile our experience of definite measurements outcomes with the linearity of TDSE.

11.10 Wigner's Formula

Remarkably, it is possible to provide probabilities for sequences of measurements on an ensemble without directly referring to the state function or to collapse, by using what is known as *Wigner's formula* for probabilities. Consider an ensemble E of n systems described by the density operator ρ and two observables A and B with eigenvalues $a_1,...,a_i,...,a_n$ and $b_1...,b_j,...,b_k$, respectively. Let us determine the probability $\Pr(a_i,b_j)$ that if we measure first A, and then B we shall get a_i and b_j. Consider the projection operators P_i^A and P_j^B related to a_i and b_j respectively. As we know,

$$\Pr(a_i) = Tr\left(\rho P_i^A\right), \tag{11.10.1}$$

and upon measurement the system will collapse onto $|\psi_i\rangle$ with density operator

$$\rho_i = |\psi_i\rangle\langle\psi_i|. \tag{11.10.2}$$

However, $Tr\left(P_i^A\rho\right) = \langle\psi_i|\rho|\psi_i\rangle$, and therefore (11.10.2) can be rewritten as

$$\rho_i = \frac{|\psi_i\rangle\langle\psi_i|\rho|\psi_i\rangle\langle\psi_i|}{Tr\left(P_i^A\rho\right)} = \frac{P_i^A\rho P_i^A}{Tr\left(P_i^A\rho\right)}. \tag{11.10.3}$$

At this point, if we measure B,

$$\Pr(b_j) = Tr\left(\rho_i P_j^B\right) \tag{11.10.4}$$

is the probability of obtaining b_j *given* that we got a_i on the first measurement. Hence,

$$\Pr(a_i,b_j) = Tr\left(\rho_i P_j^B\right)Tr\left(\rho P_i^A\right) = Tr\left(\frac{P_i^A\rho P_i^A P_j^B}{Tr\left(P_i^A\rho\right)}\right)Tr\left(\rho P_i^A\right), \tag{11.10.5}$$

that is,

$$\Pr(a_i,b_j) = Tr\left(P_i^A\rho P_i^A P_j^B\right). \tag{11.10.6}$$

Formula (11.10.6) can be generalized to more than two measurement returns separated by finite time intervals and it is a simplified version of Wigner's formula.[20] Now one could take Wigner's formula as primitive and use it to predict the returns of any sequence of measurements one wishes without having directly to appeal to the projection postulate or even to TDSE. Nevertheless, if one maintains that quantum states are represented by state vectors and that the eigenvalue-eigenvector principle holds, every time a new measurement return occurs, one must assume that collapse has taken place, even if Wigner's formula is silent on it. Indeed, Wigner himself never gave up the idea of collapse, as we shall see. However, one might abandon the idea of state vector altogether and just work with density operators, concluding that there is no abrupt non-linear collapse simply because there is nothing to do the collapsing, as it were. Such an attitude could be justified by noting that even in standard quantum mechanics quantum states are not measurable anyway. In other words, one could view collapse as a mathematical artifact associated to the (avoidable) introduction of state vectors, a sort of mathematical and historical curiosity.

However, the price to be paid for this maneuver is an increase in the lack of perspicuity of quantum mechanics. For the collapse postulate explains not only why the system comes out of superposition but also why quickly repeated measurements have the same result; indeed, this was one of the main reasons for its introduction. By contrast, Wigner's formula does not explain why repeated measurements have the same results. Of course, if one looks at quantum mechanics as a mere algorithm to make correct predictions, Wigner's formula will do just fine; but then one would not be too bothered by the measurement problem in the first place: collapse works well, even if we cannot say much about what makes measurement such a peculiar affair to disrupt TDSE.

[20] See appendix 4.

Exercises

Exercise 11.1

1. Prove that $P_A{}^2 = P_A$.

2. Prove that $Tr(|A\rangle\langle A|) = 1$. [Hint. The arguments of Tr can be rotated cyclically: the rightmost can be moved to the leftmost position and vice versa; for example, $Tr(ABC) = Tr(CBA) = Tr(BAC)$. Use this to prove $Tr(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle$ and then apply this result to obtain what we want].

3. True or false: if $|\psi_1\rangle, \ldots, |\psi_n\rangle$ span the space, then $\sum_i P_i = 1$.

Exercise 11.2

1. Prove that the density operator is Hermitian. [Hint: We need to show that $\langle\Xi|\rho|\Phi\rangle^* = \langle\Phi|\rho|\Xi\rangle$. From the definition of density operator, we have that $\langle\Xi|\rho|\Phi\rangle = \sum_i p_i \langle\Xi|\Psi_i\rangle\langle\Psi_i|\Phi\rangle$. Now construct the complex conjugate of the summation's argument and from that obtain $\langle\Phi|\rho|\Xi\rangle^*$.]

2. Construct the density operator for the pure system $|\Psi\rangle = \frac{1}{3}\begin{pmatrix} 2-i \\ 2 \end{pmatrix}$ and determine $< S_z >$ and $< S_x >$.

3. There is a relatively simple way of constructing the density matrix in the basis $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ for a pure system $|\Psi\rangle = \sum_n c_n |\psi_n\rangle$: simply set the elements of the matrix as $\rho_{i,j} = c_j{}^* c_i$. Construct the density matrix for the system $|\Psi\rangle = \frac{1}{\sqrt{15}}\begin{pmatrix} 3+i \\ 2-i \end{pmatrix}$.

4. Construct the density matrix in $H = H_1 \otimes H_2$ corresponding to (11.2.13).

Exercise 11.3

Consider two entangled particles 1+2 in state $|\Psi'\rangle = \alpha|e_1 d_1\rangle - \beta|e_2 d_2\rangle$. Determine the density operator for the whole system and the reduced density operators for 1 and 2.

Answers to the Exercises

Exercise 11.1

1. $P_A{}^2 = |A\rangle\langle A|A\rangle\langle A|$, and since $\langle A|A\rangle = 1$, $P_A{}^2 = |A\rangle\langle A|A\rangle\langle A| = |A\rangle\langle A| = P_A$.

2. $Tr(|\psi\rangle\langle\phi|) = Tr\langle\phi|\psi\rangle = \langle\phi|\psi\rangle$. Applying this result the projection operator we obtain

$Tr(|A\rangle\langle A|) = \langle A|A\rangle = 1$.

3. True, as it directly follows from (11.1.3).

Exercise 11.2

1. $\langle\Xi|\rho|\Phi\rangle = \sum_i p_i \langle\Xi|\Psi_i\rangle\langle\Psi_i|\Phi\rangle$, and consequently

$$\langle\Xi|\rho|\Phi\rangle^* = \left(\sum_i p_i \langle\Xi|\Psi_i\rangle\langle\Psi_i|\Phi\rangle\right)^* = \sum_i p_i \langle\Phi|\Psi_i\rangle\langle\Psi_i|\Xi\rangle = \langle\Phi|\rho|\Xi\rangle.$$

2. $\rho = \frac{1}{9}\begin{pmatrix} 2-i \\ 2 \end{pmatrix}\begin{pmatrix} 2+i & 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 5 & 4-2i \\ 4+2i & 4 \end{pmatrix}$. Hence,

$$< S_z > = \frac{\hbar}{18} Tr\left[\begin{pmatrix} 5 & 4-2i \\ 4+2i & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \frac{\hbar}{18}(5-4) = \frac{\hbar}{18};$$

$$< S_x > = \frac{\hbar}{18} Tr\left[\begin{pmatrix} 5 & 4-21 \\ 4+2i & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \frac{4}{9}\hbar.$$

3. First, let us express the state vector explicitly in terms of the basis vectors:

$|\Psi\rangle = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \frac{3+i}{\sqrt{15}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{2-i}{\sqrt{15}}\begin{pmatrix}0\\1\end{pmatrix}$. Then, in that basis**,**

$$\rho = \begin{pmatrix} |c_1|^2 & c_1^* c_2 \\ c_1 c_2^* & |c_2|^2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1-i}{3} \\ \frac{7-i}{15} & \frac{1}{3} \end{pmatrix}.$$

4. $\rho = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$. Note that this matrix is 2x2 while it should be 4x4 because it operates in a 4-dimensional space. However, we can compress the notation by eliminating all the columns and rows containing only elements equal to zero.

Exercise 11.3

$$\rho_{\Psi'} = |\alpha|^2 \big(|e_1\rangle \otimes |d_1\rangle\big)\big(\langle e_1| \otimes \langle d_1|\big) - \alpha\beta^* \big(|e_1\rangle \otimes |d_1\rangle\big)\big(\langle e_2| \otimes \langle d_2|\big) - \alpha^*\beta \big(|e_2\rangle \otimes |d_2\rangle\big)\big(\langle e_1| \otimes \langle d_1|\big) +$$
$$|\beta|^2 \big(|e_2\rangle \otimes |d_2\rangle\big)\big(\langle e_2| \otimes \langle d_2|\big),$$

that is,

$$\rho_{\Psi'} = |\alpha|^2 \big(|e_1\rangle\langle e_1| \otimes |d_1\rangle\langle d_1|\big) - \alpha\beta^* \big(|e_1\rangle\langle e_2| \otimes |d_1\rangle\langle d_2|\big) - \alpha^*\beta \big(|e_2\rangle\langle e_1| \otimes |d_2\rangle\langle d_1|\big) +$$
$$|\beta|^2 \big(|e_2\rangle\langle e_2| \otimes |d_2\rangle\langle d_2|\big).$$

Then, $\rho_1 = |\alpha|^2 |e_1\rangle\langle e_1| + |\beta|^2 |e_2\rangle\langle e_2|$, and $\rho_2 = |\alpha|^2 |d_1\rangle\langle d_1| + |\beta|^2 |d_2\rangle\langle d_2|$.